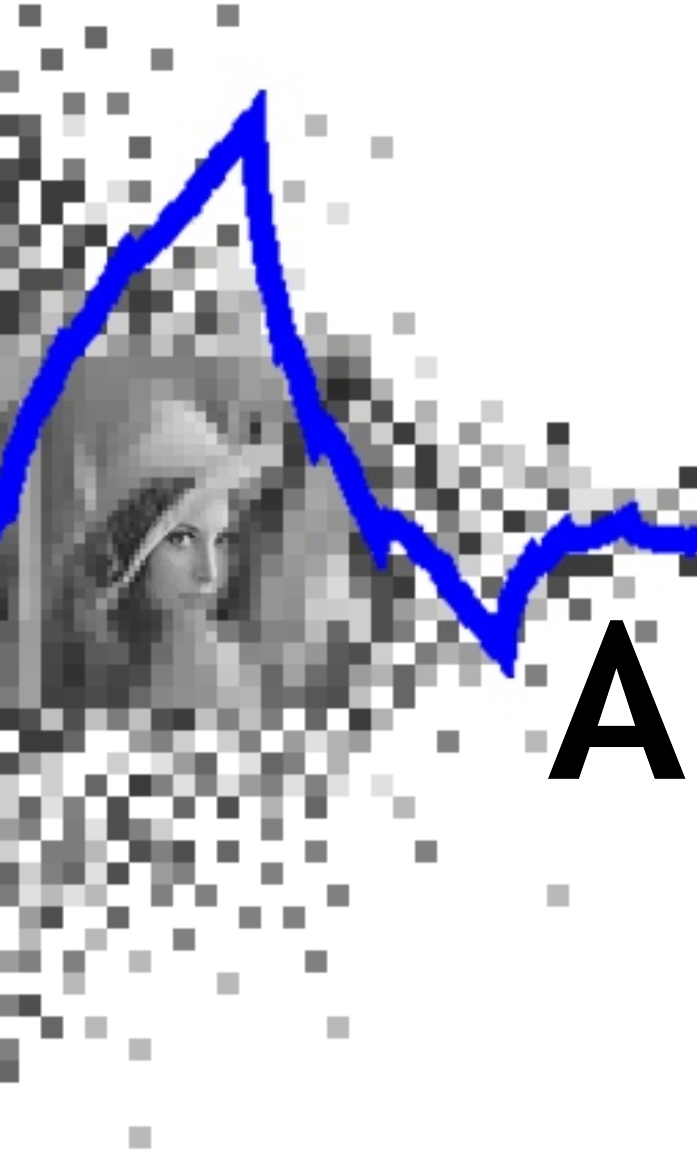


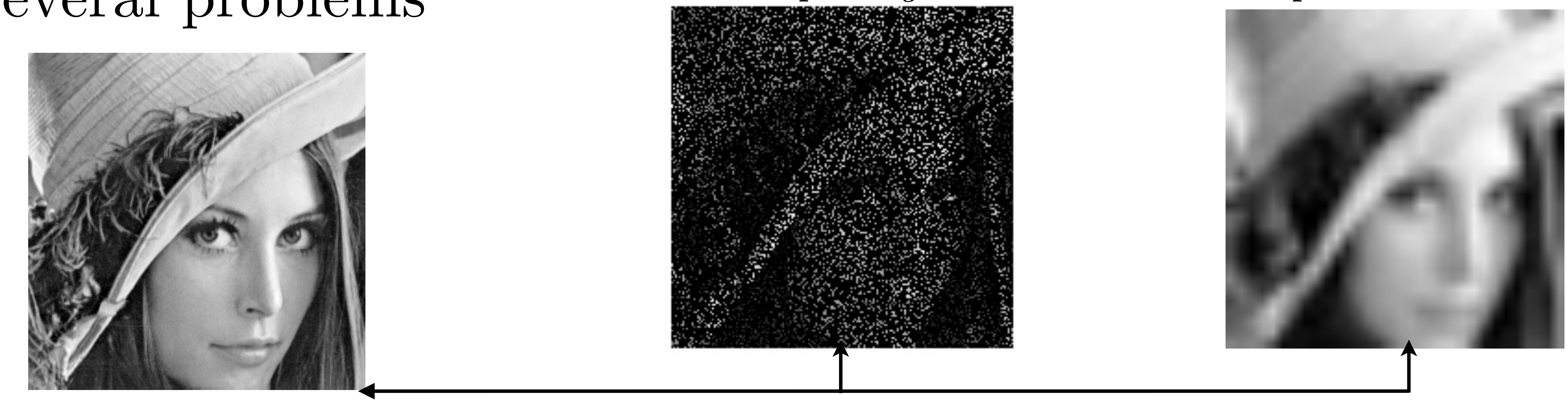
Robust Sparse Analysis Recovery

Samuel Vaiter

Inverse Problems

Inverse Problems

Several problems



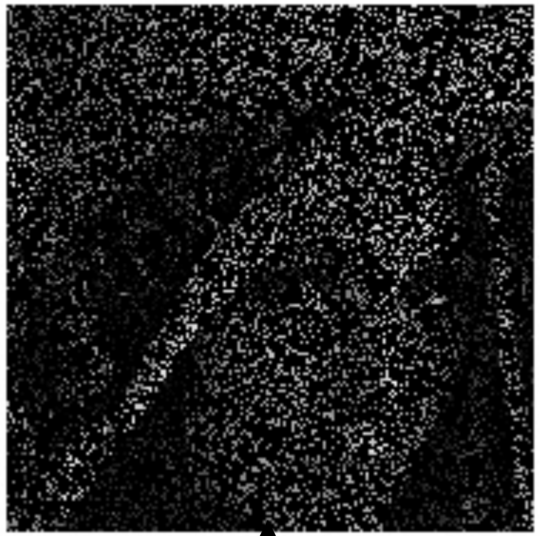
Inpaiting

Super-resolution

Inverse Problems

Several problems

Inpaiting



Super-resolution

One model

Observations ⟶ $y = \Phi x_0 + w$ ⟵ Noise

Operator ↗ ↖ Unknown signal

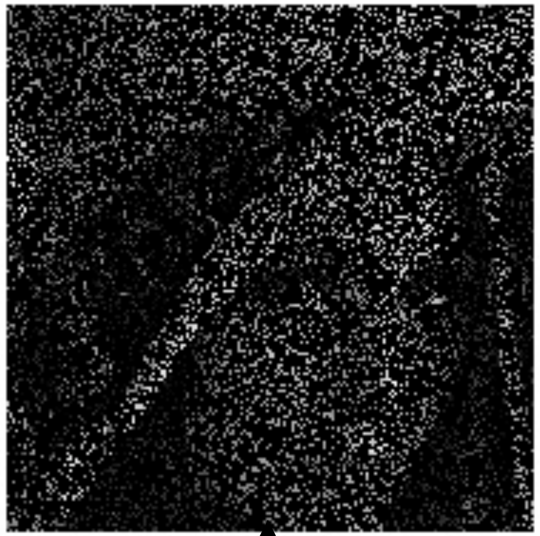
Linear hypothesis

Φ ill-posed

Inverse Problems

Several problems

Inpaiting



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Regularization

$$x^\star \in \underset{x\in\mathbb{R}^N}{\operatorname{argmin}} \underbrace{\boxed{\frac{1}{2}\|y-\Phi x\|_2^2}}_{\text{Data fitting}} + \underbrace{\boxed{\lambda J(x)}}_{\text{Regularity}}$$

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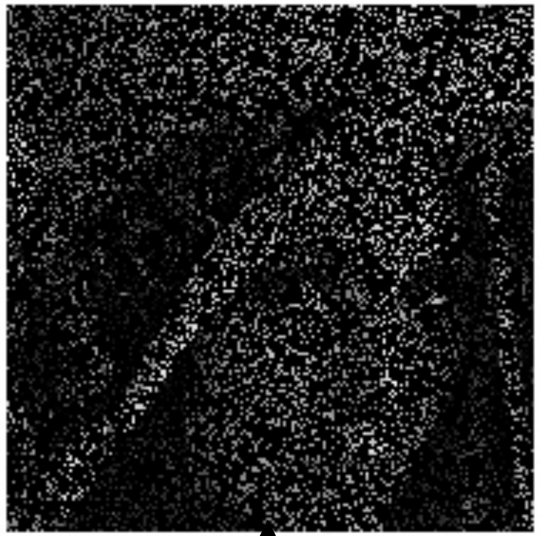

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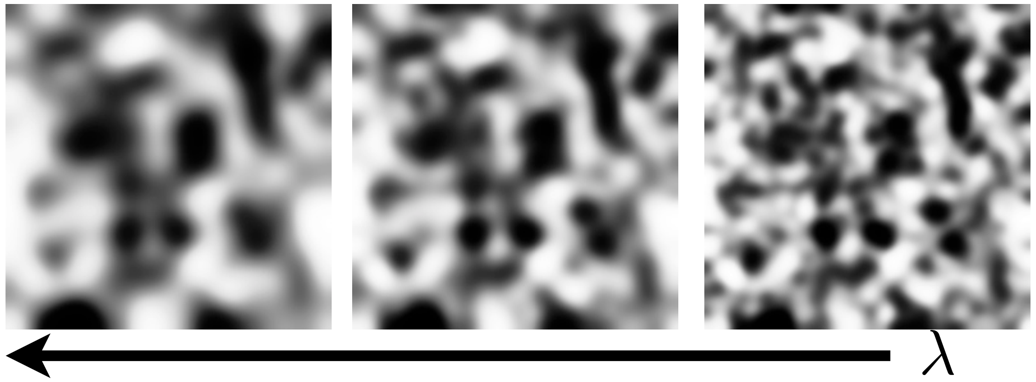
Image Priors

Sobolev

$$J(x) = \frac{1}{2}\int \|\nabla x\|^2$$



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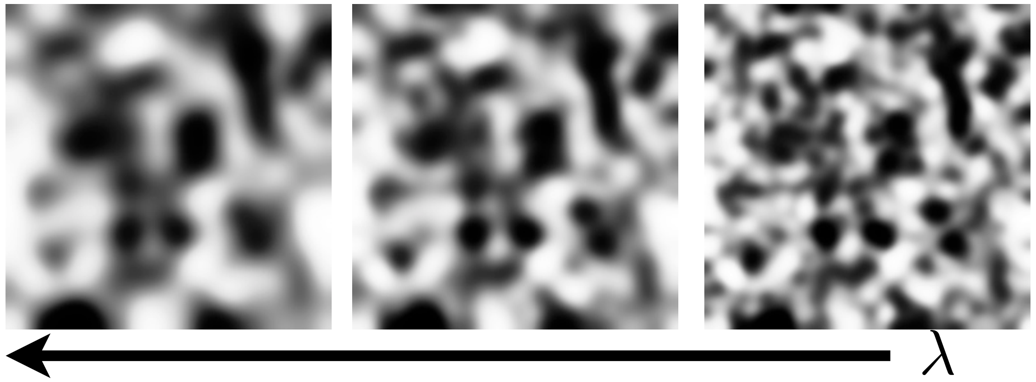

Total variation

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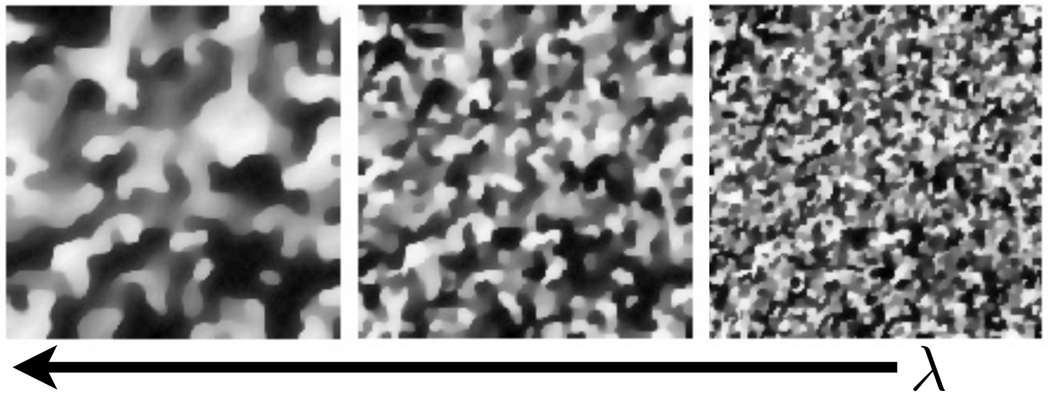

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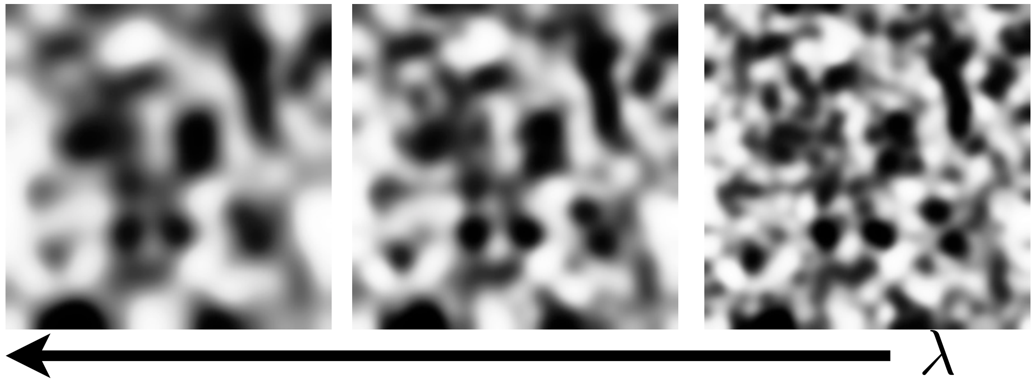

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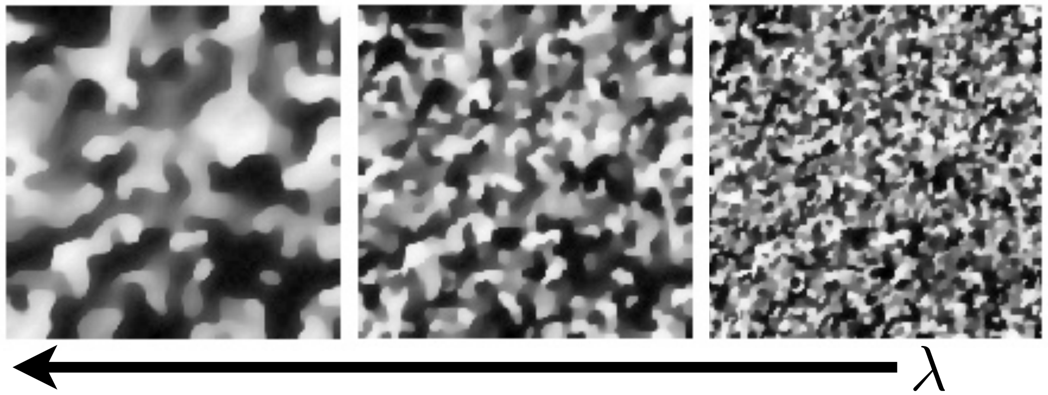
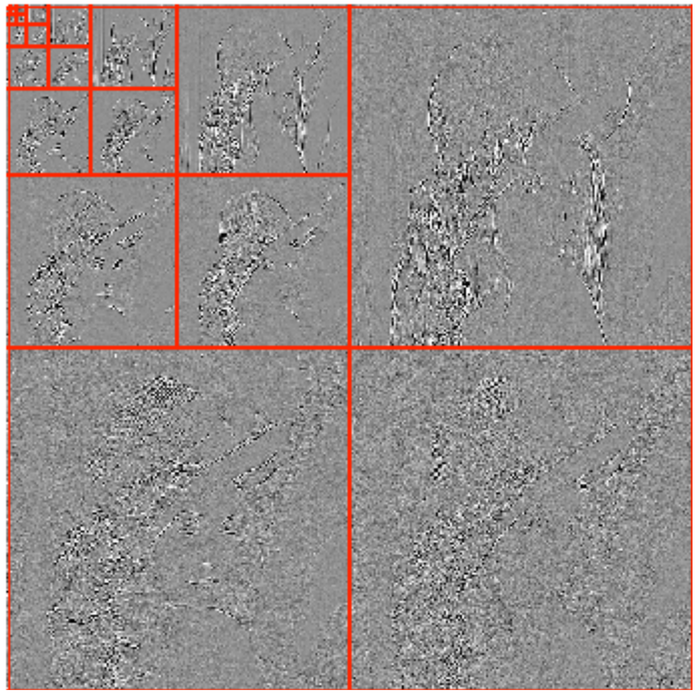


Wavelet sparsity

$$J(x) = |\,\{i \setminus \langle x,\, \psi_i \rangle \neq 0\}\,|$$

(ideal prior)



Overview

- Analysis vs. Synthesis Regularization
- Local Parameterization of Analysis Regularization
- Identifiability and Stability
- Numerical Evaluation
- Perspectives

Dictionary

Redundant dictionary of $\mathbb{R}^N : \{d_i\}_{i=0}^{P-1}, P \geqslant N$

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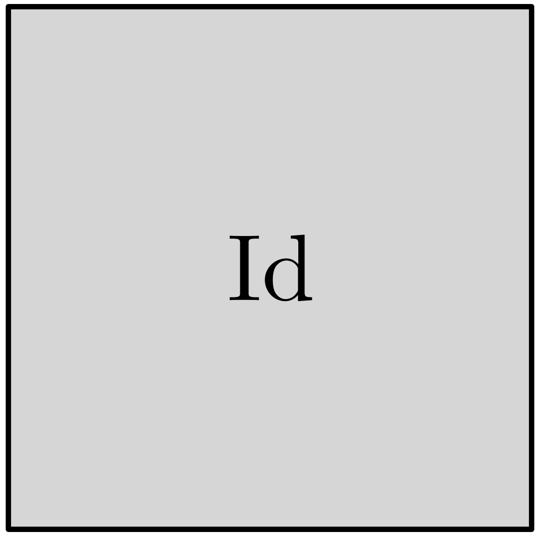

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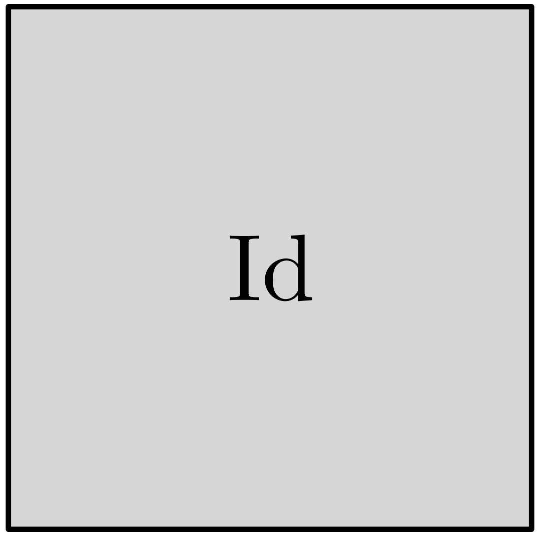

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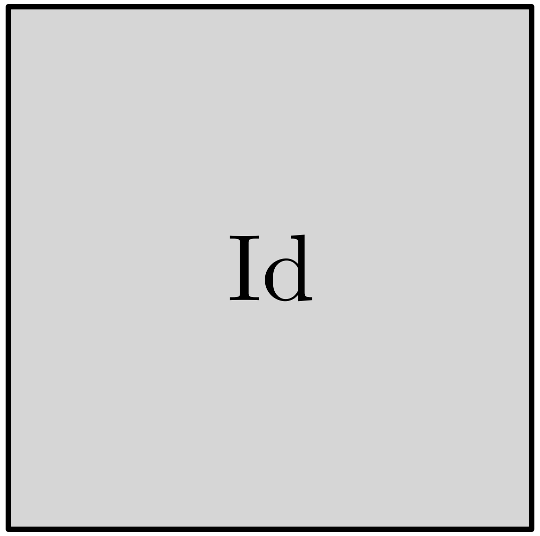

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finite difference operator

$$\Omega_{\mathrm{DIF}}^{n}$$

$$\begin{pmatrix} -1 & & & 0 \\ +1 & -1 & & \\ & +1 & \ddots & \\ & & \ddots & -1 \\ 0 & & & +1 \end{pmatrix}$$

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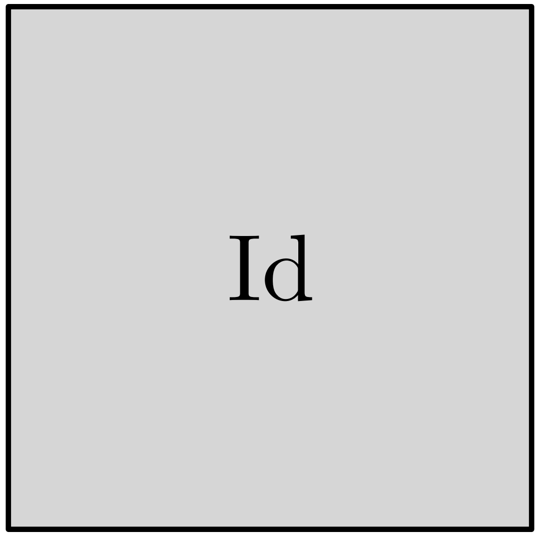

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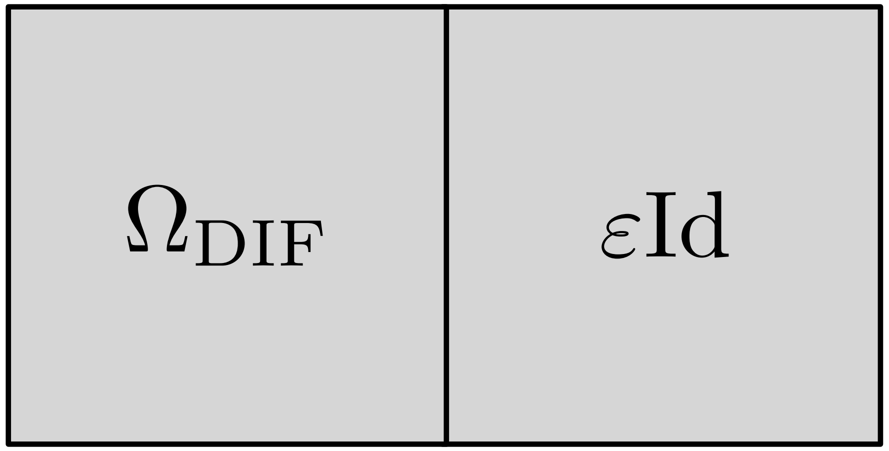


Ω_{DIF}

εId

Analysis versus Synthesis

Two point of view

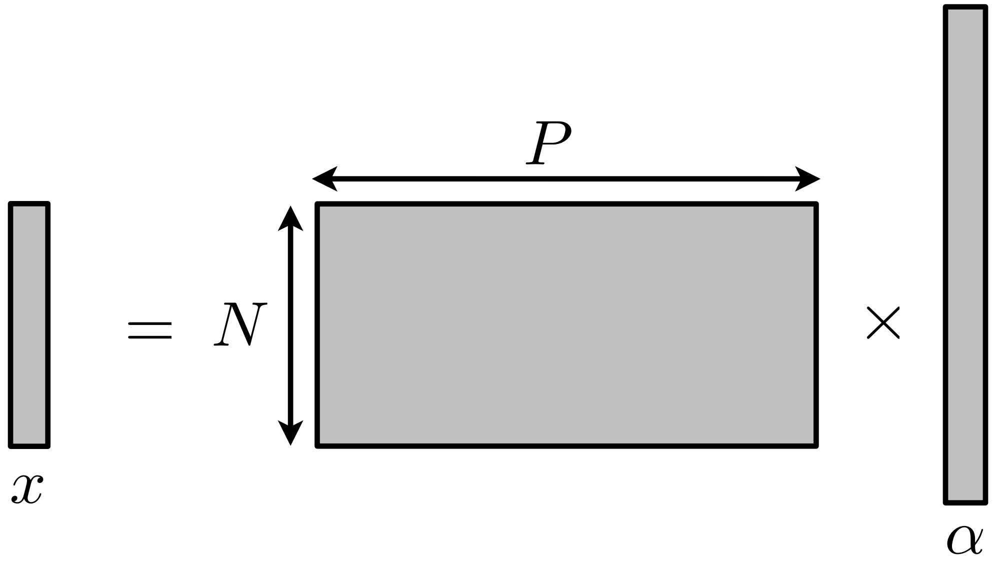
Analysis versus Synthesis

Two point of view

$$x = D\alpha$$

Synthesis

$\longrightarrow$ non-unique if $P > N$



"Generate" x

Analysis versus Synthesis

Two point of view

$x = D\alpha$

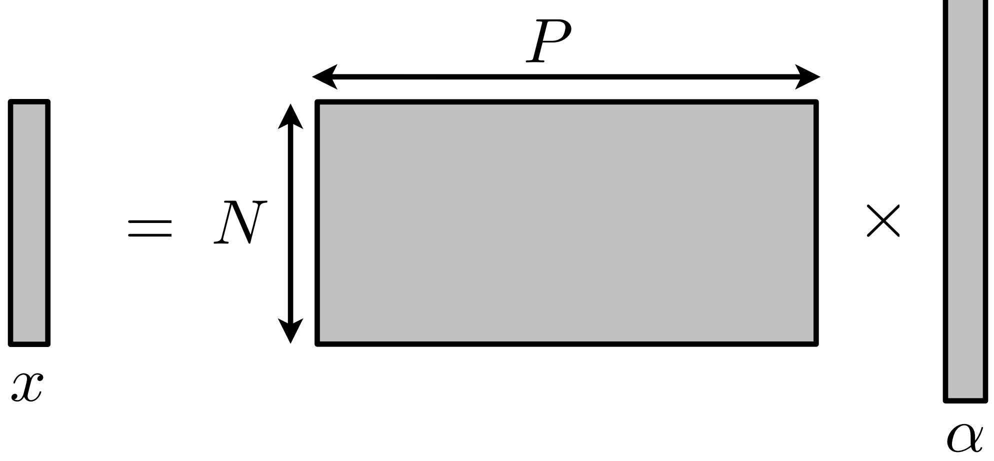
Synthesis

OR ?

$D^* x = \alpha$

Analysis

$\longrightarrow$ non-unique if $P > N$



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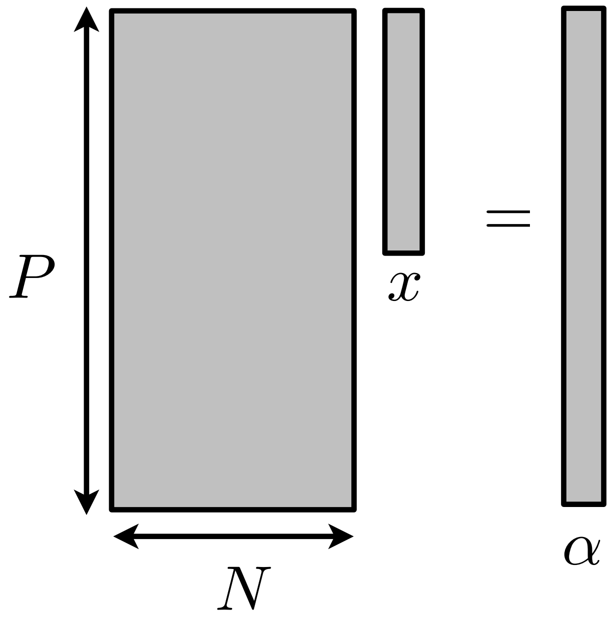

“Analyze” x

A Bird's Eye View of Sparsity

"Ideal" sparsity prior: $J_0(\alpha) = |\,\{i \setminus \alpha_i \neq 0\}\,|$

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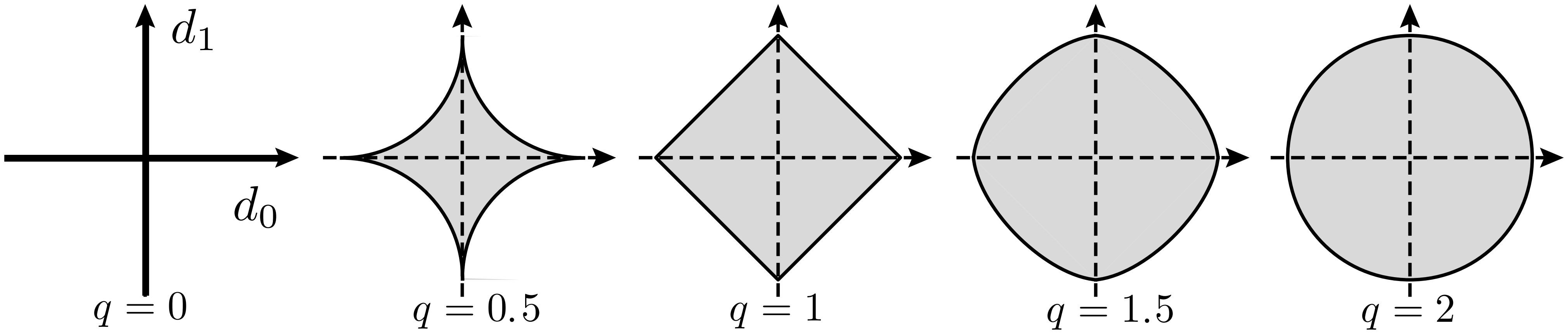

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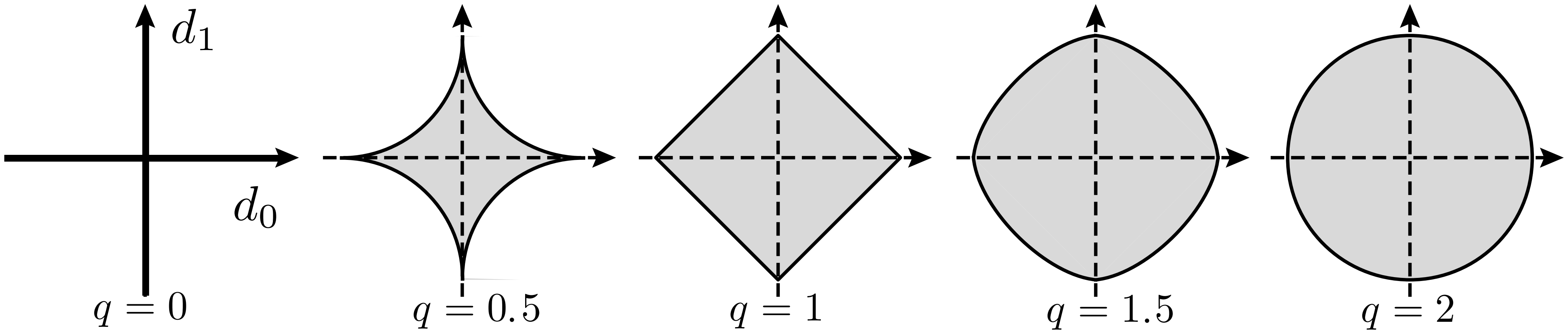

ℓ^1 norm: *convexification* of ℓ^0 prior

Sparse Regularizations

Synthesis

$$\underset{\alpha \in \mathbb{R}^Q}{\operatorname{argmin}} \ \frac{1}{2}\|y - \Psi\alpha\|_2^2 + \lambda\|\alpha\|_1$$

$$\Psi = \Phi D \qquad x = D\alpha$$

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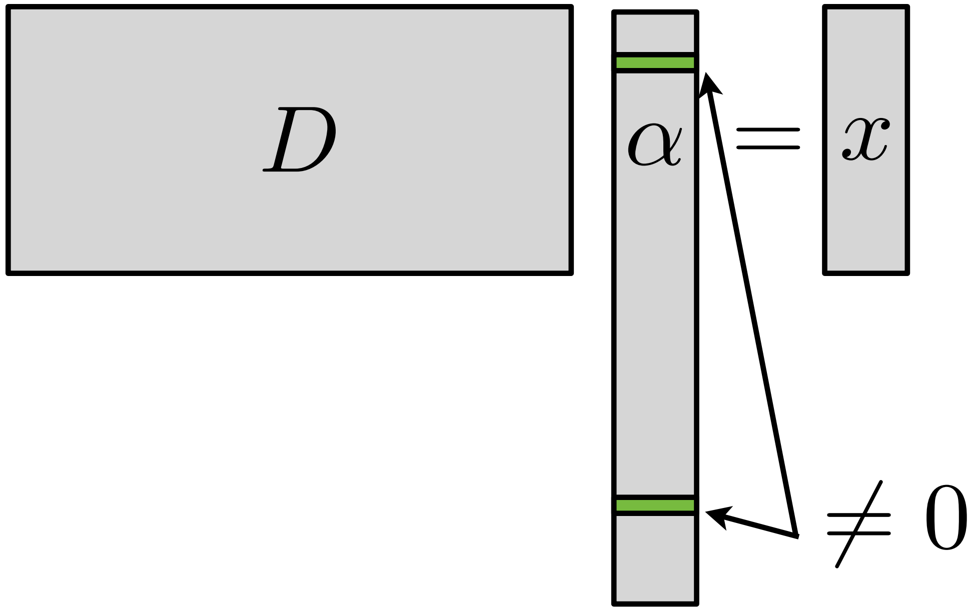

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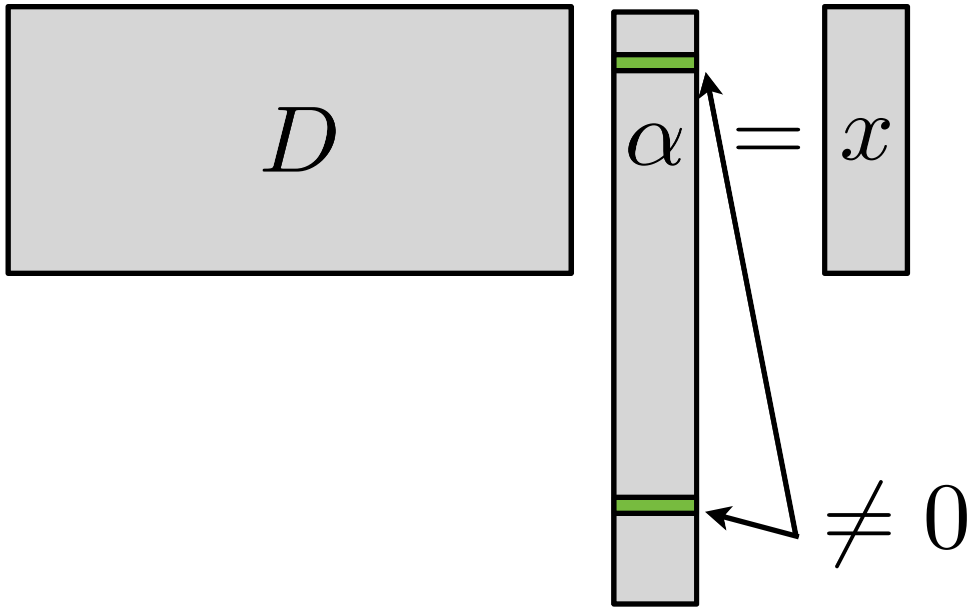

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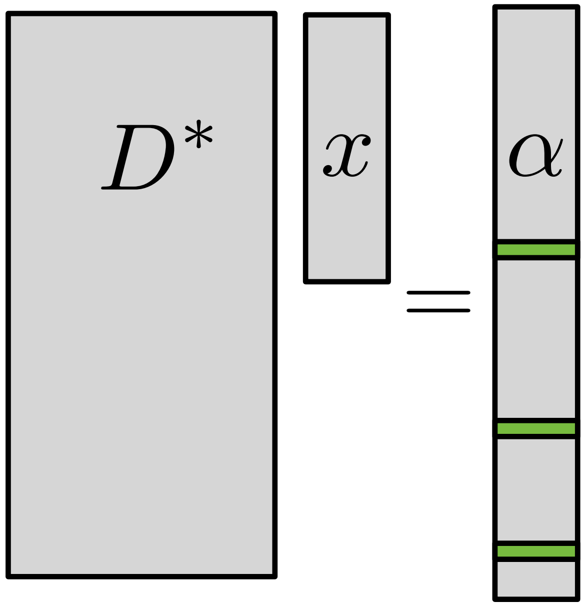

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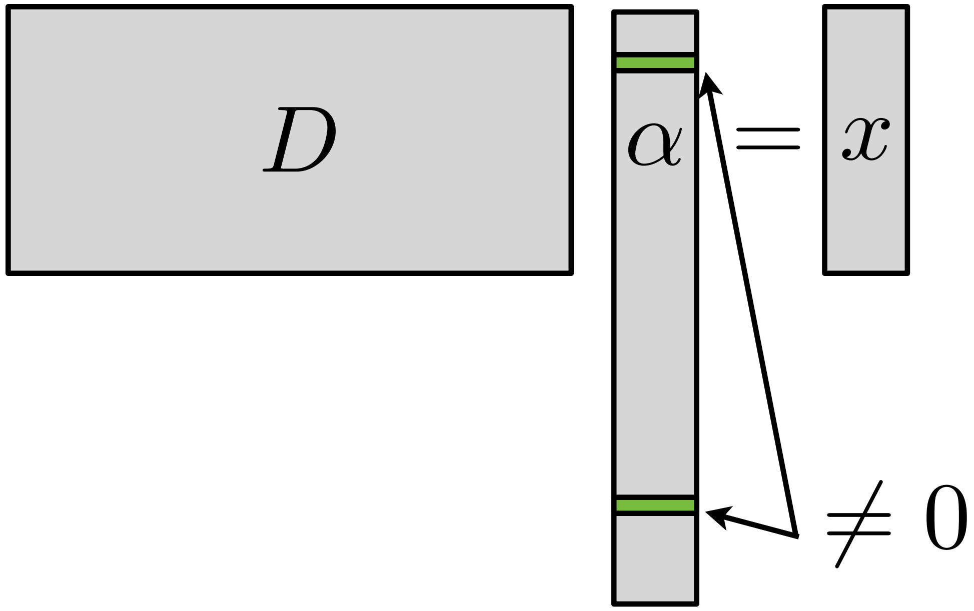

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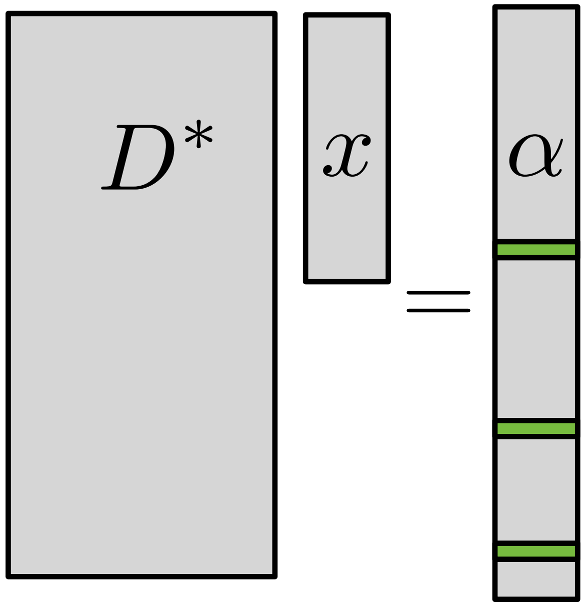

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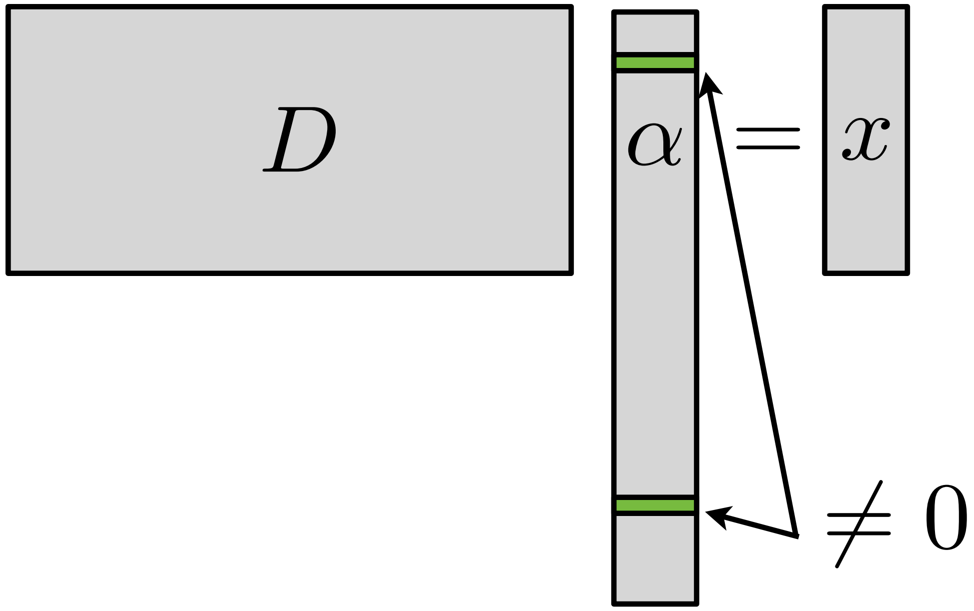

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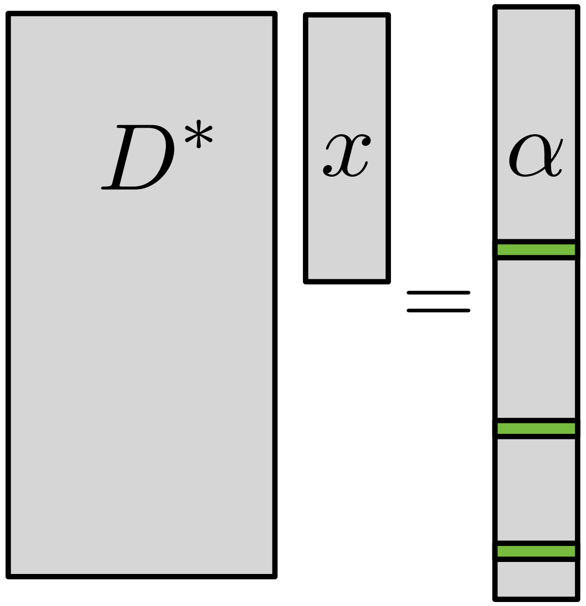

Correlation of $x^\star$ and D sparse

Support and Signal Model

$$x^\star \in \underset{x \in \mathbb{R}^N}{\operatorname{argmin}} \frac{1}{2}\|y - \Phi x\|_2^2 + \lambda \|D^* x\|_1 \qquad \mathcal{P}(y, \lambda)$$

$$I = \operatorname{supp}(D^* x^\star), J = I^c$$

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Signal model : *"Union of subspace"*

$$\Theta = \bigcup_{k \in \{1 \dots P\}} \Theta_k \quad \text{where} \quad \Theta_k = \{\mathcal{G}_J \setminus \dim \mathcal{G}_J = k\}$$

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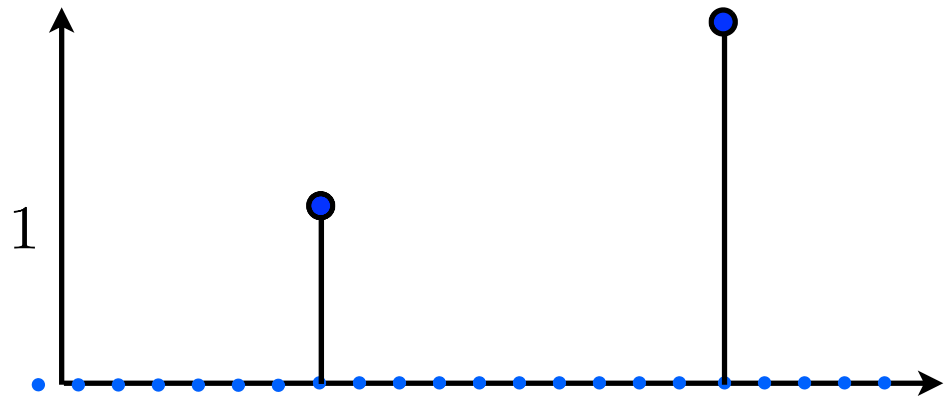
Hypothesis: $\operatorname{Ker} \Phi \cap \operatorname{Ker} D^* = \{0\}$

Examples of Signal Model

Identity

Θ_k : k-sparse signals



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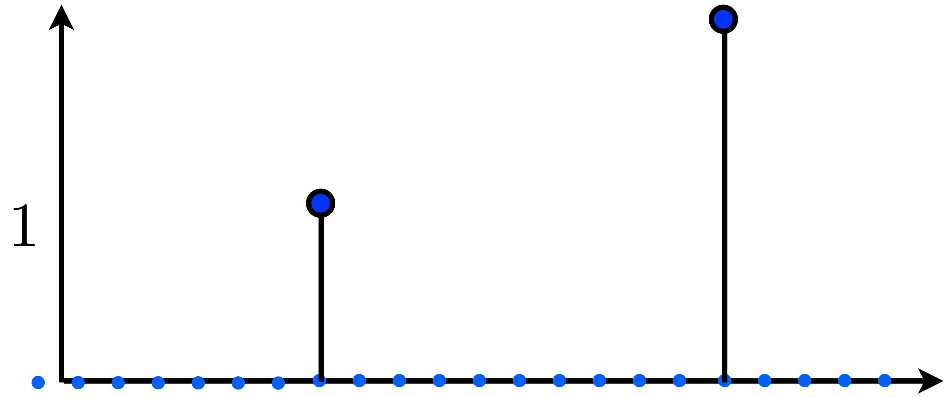
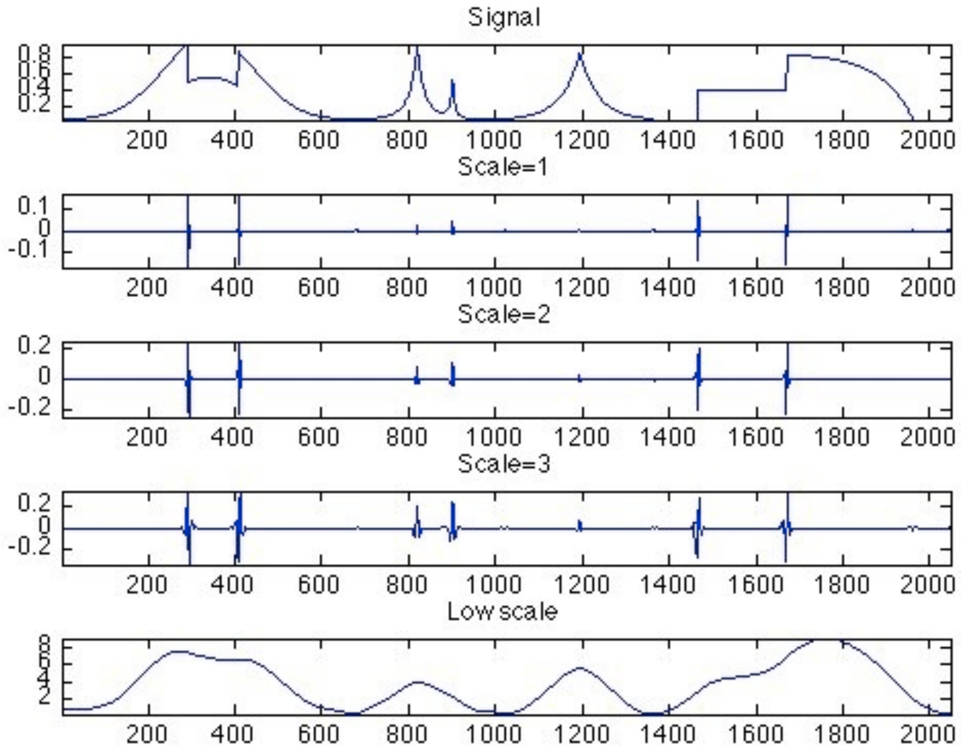


shift invariant wavelet frame



Credit to G. Peyré

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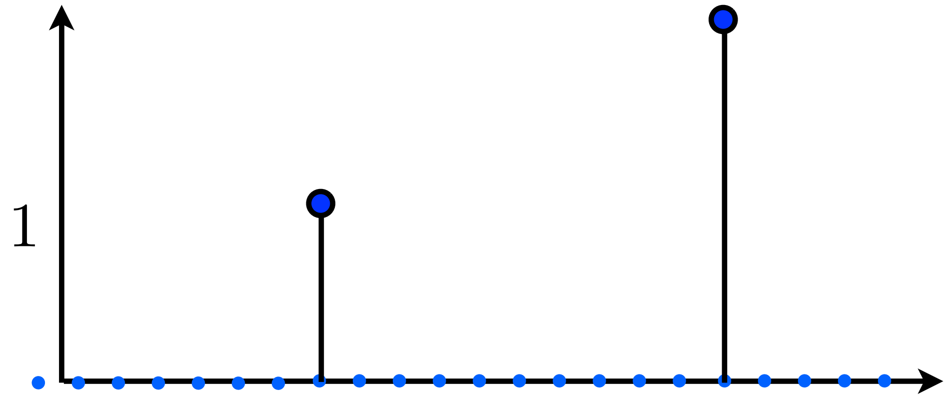

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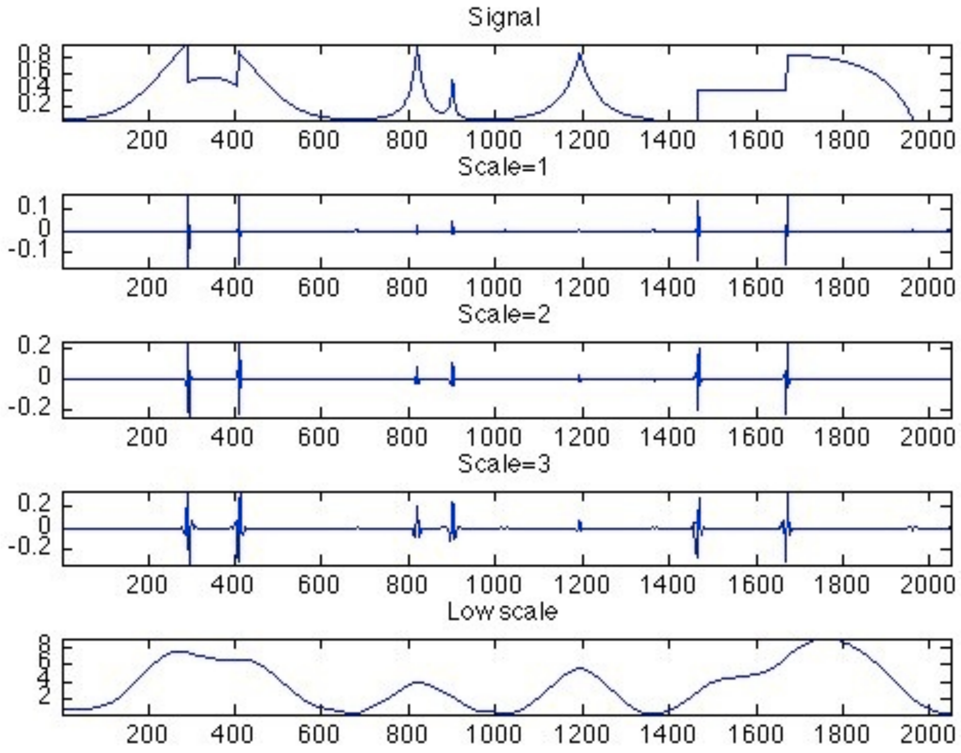

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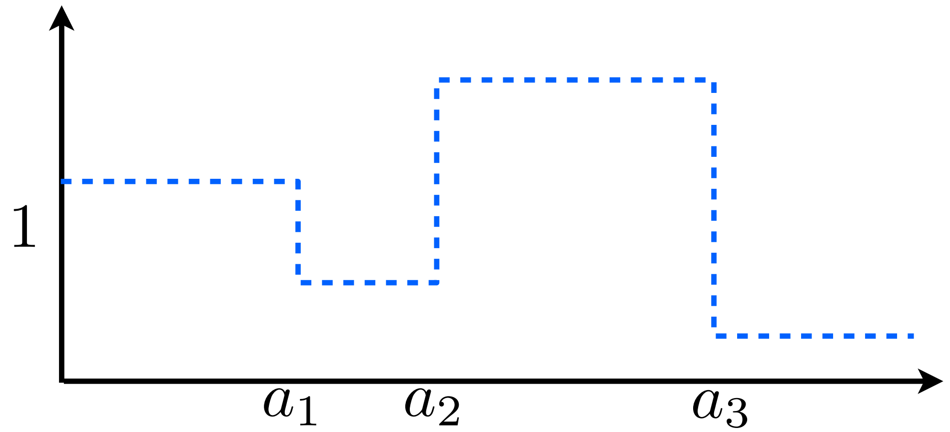
finite difference operator

Θ_k : piecewise constant signals with $k-1$ steps



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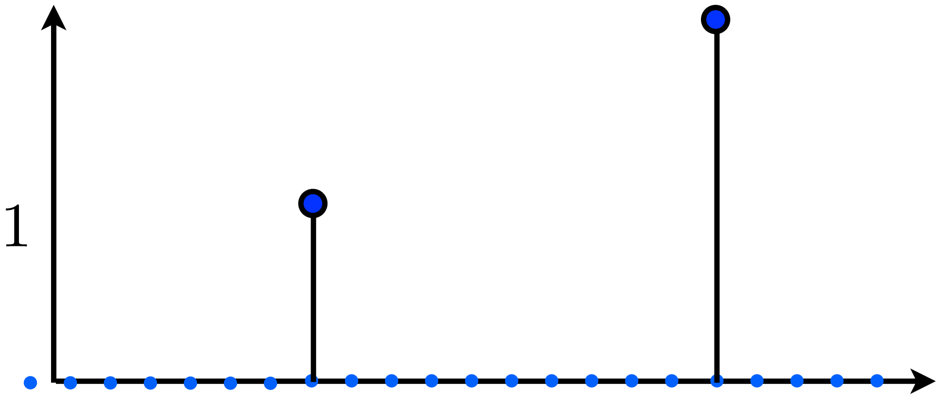

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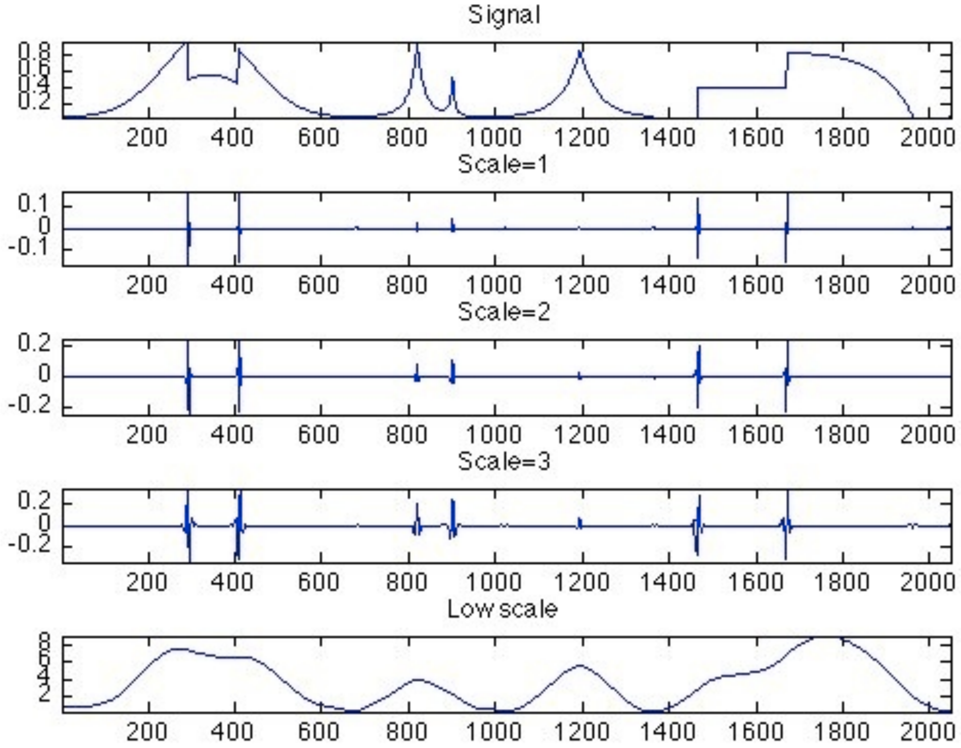

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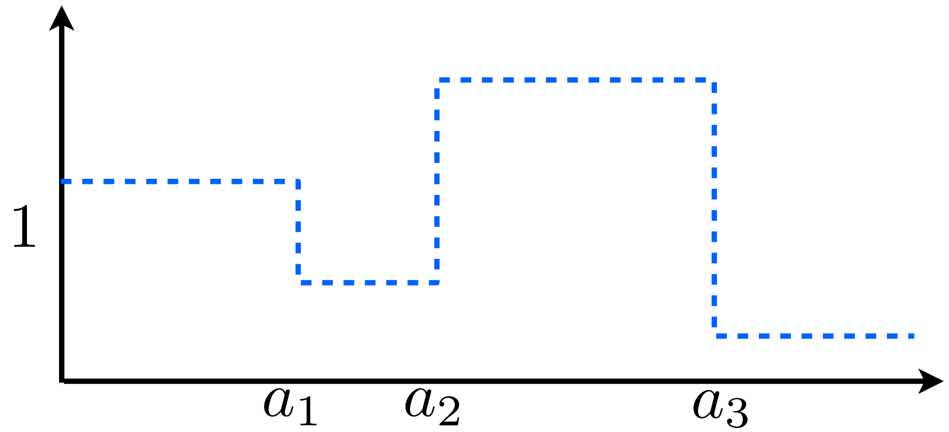
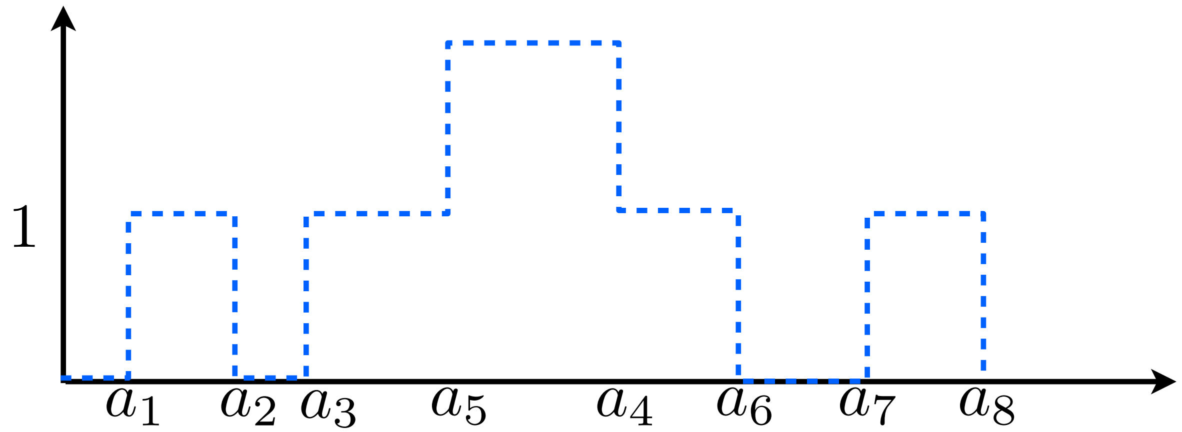


fussed lasso

Θ_k : sum of k interval characteristic functions



Remember !

Synthesis

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Analysis

$$x^\star = \underset{x \in \mathbb{R}^N}{\operatorname{argmin}} \frac{1}{2}\|y - \Phi x\|_2^2 + \lambda\|D^* x\|_1 \qquad \mathcal{P}(y,\lambda)$$

$$\Big\downarrow \; \lambda \to 0$$

$$x^\star = \underset{\Phi x = y}{\operatorname{argmin}} \|D^* x\|_1 \qquad \mathcal{P}(y,0)$$

Toward a Better Understanding

Local behavior ?

Properties of $x^\star$ solution of $\mathcal{P}(y,\lambda)$ as a function of y

Toward a Better Understanding

Local behavior ?

$$\text{Properties of } x^\star \text{ solution of } \mathcal{P}(y, \lambda) \text{ as a function of } y$$

Noiseless identifiability ?

$$\text{Is } x_0 \text{ the unique solution of } \mathcal{P}(\Phi x_0, 0) \text{ ?}$$

Toward a Better Understanding

Local behavior ?

Properties of $x^\star$ solution of $\mathcal{P}(y,\lambda)$ as a function of y

Noiseless identifiability ?

Is x_0 the unique solution of $\mathcal{P}(\Phi x_0, 0)$?

Noise robustness ?

What can we say about $\|x^\star - x_0\|$ for noisy observations ?

From Synthesis to Analysis Results

— Previous works in synthesis

[Fuchs, Tropp, Dossal]: address these questions

From Synthesis to Analysis Results

— Previous works in synthesis

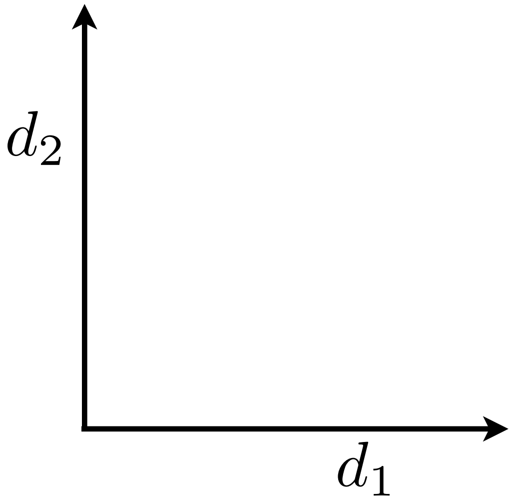
[Fuchs, Tropp, Dossal]: address these questions

— Similar problem but much more difficulties in analysis

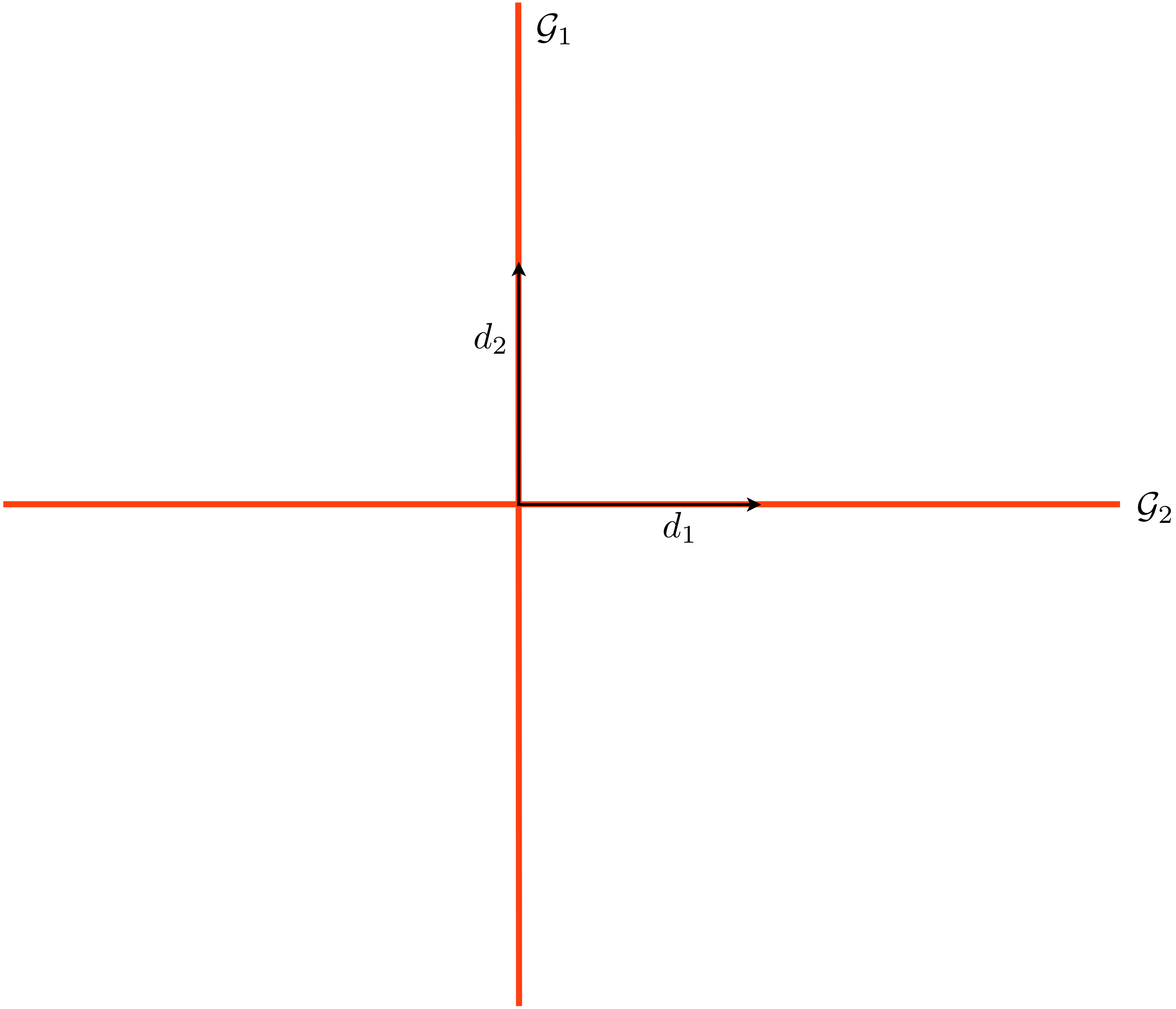
Geometry of the problem ?

From Synthesis to Analysis Results

From Synthesis to Analysis Results



From Synthesis to Analysis Results



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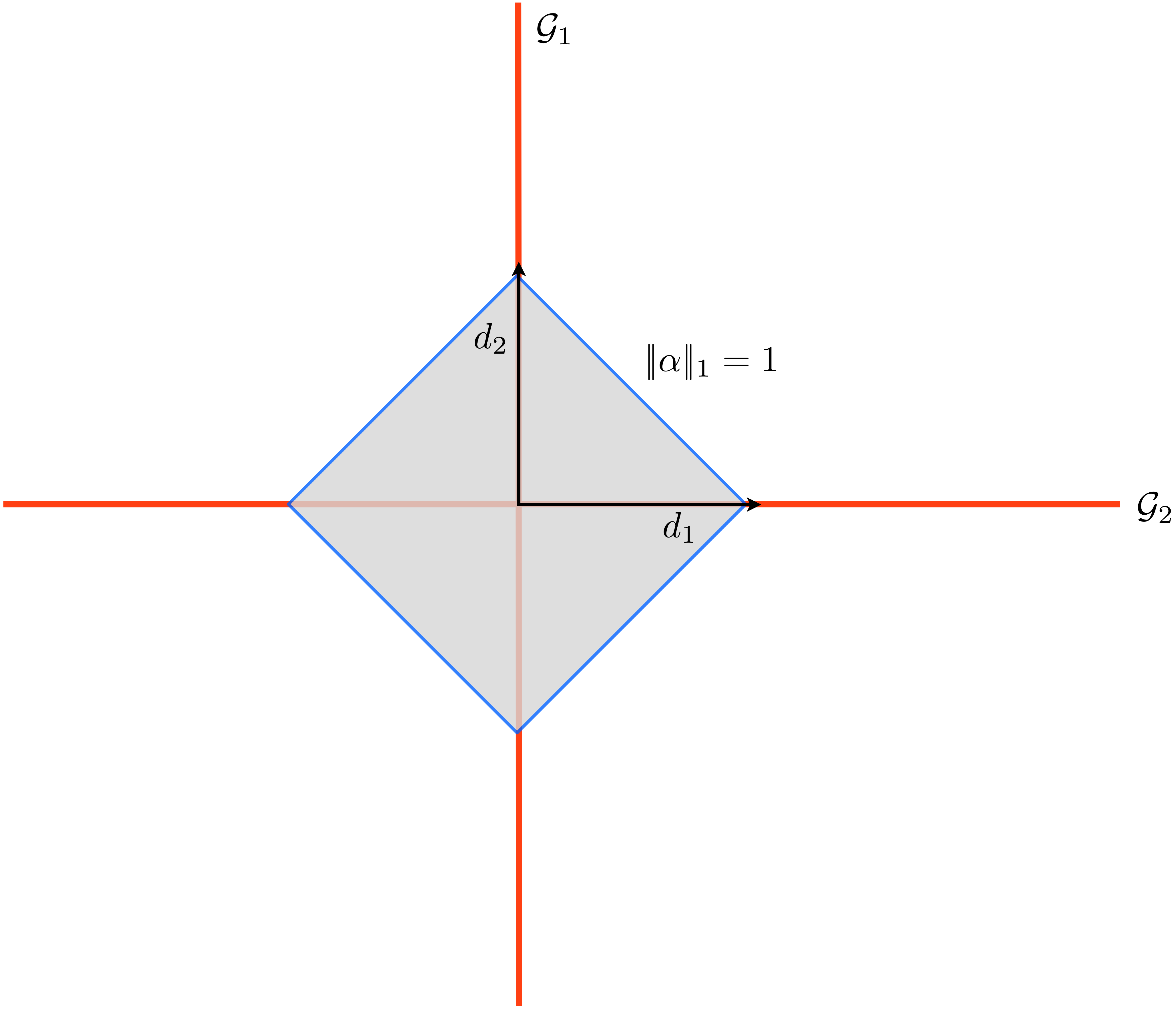

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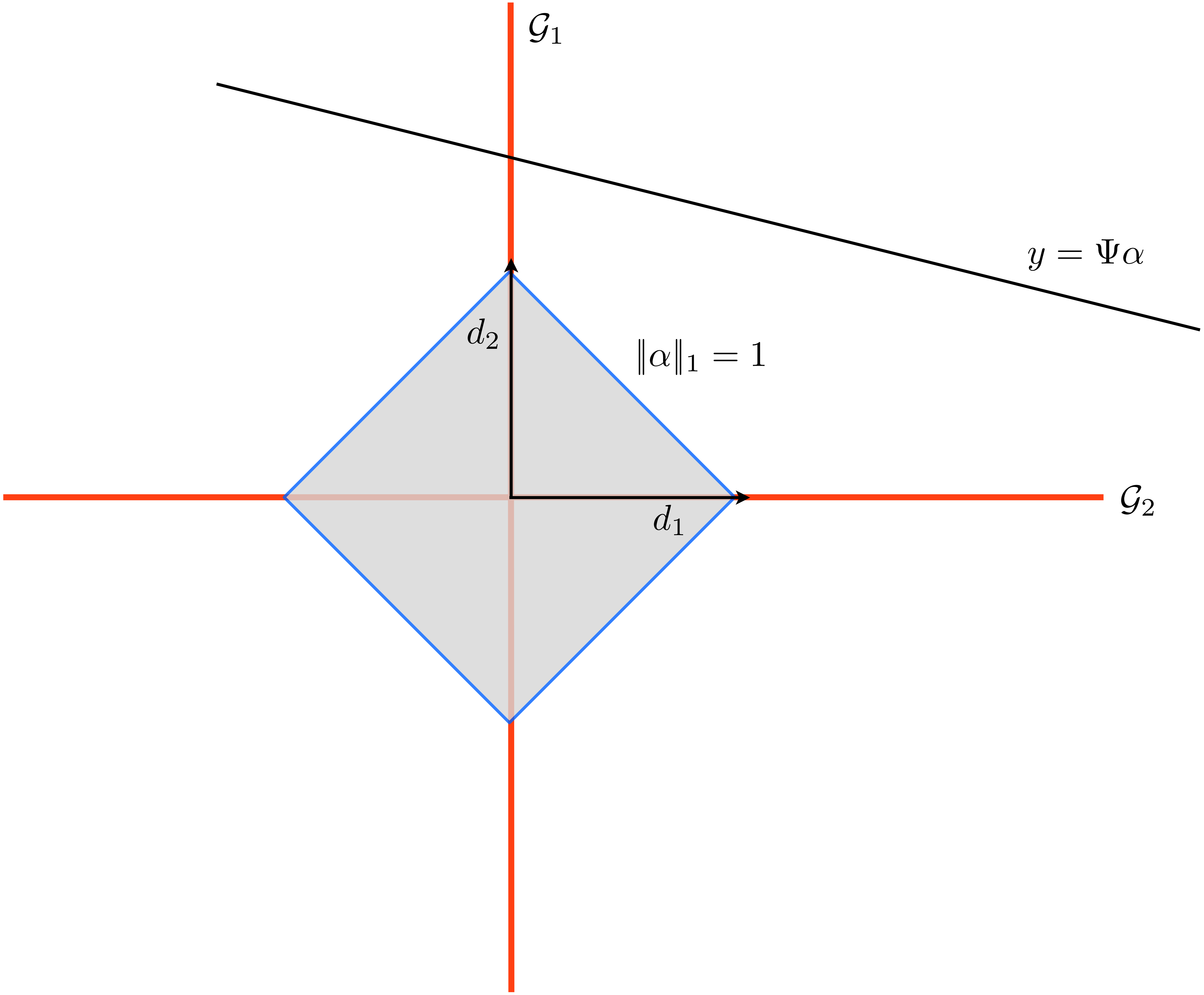

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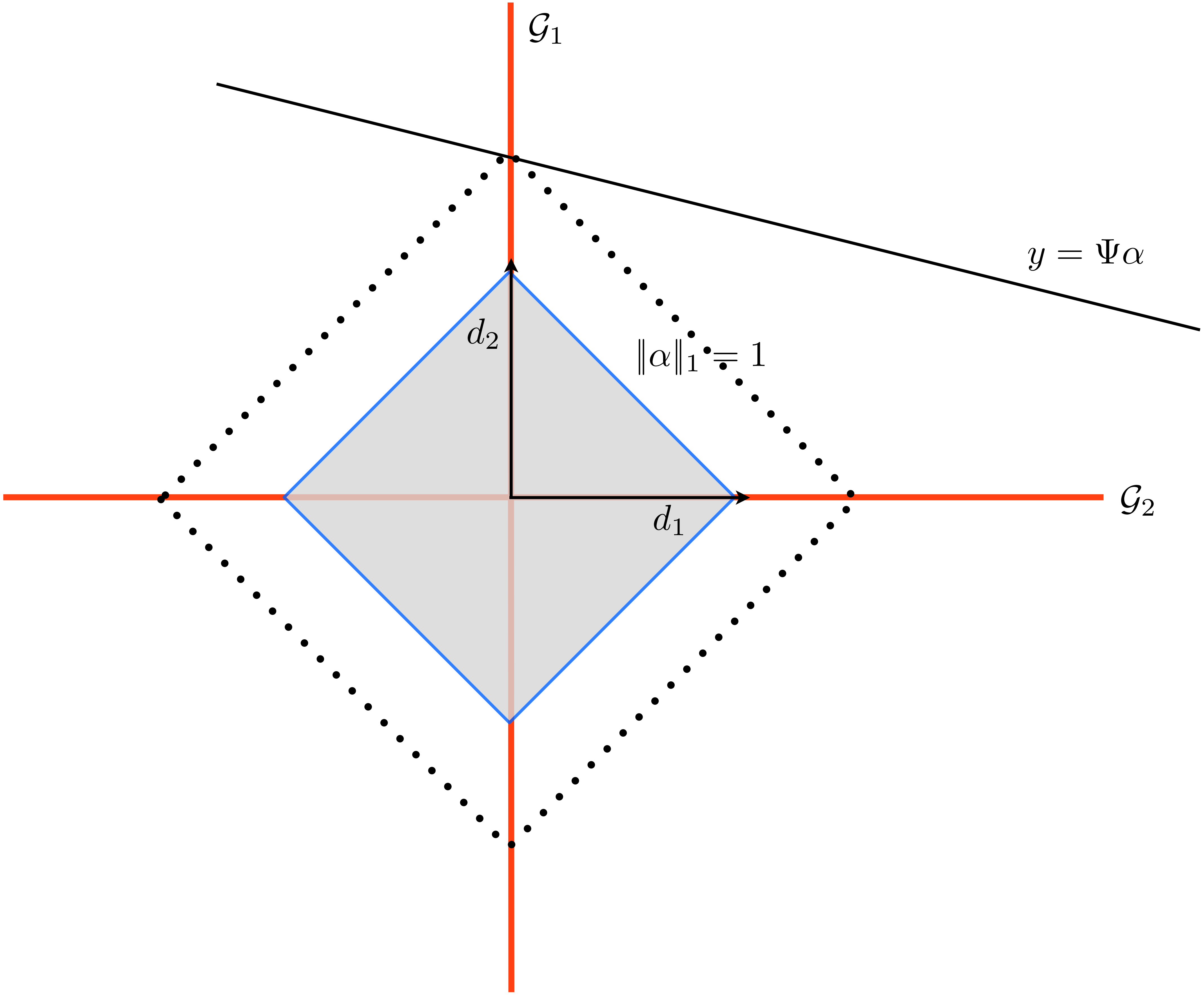
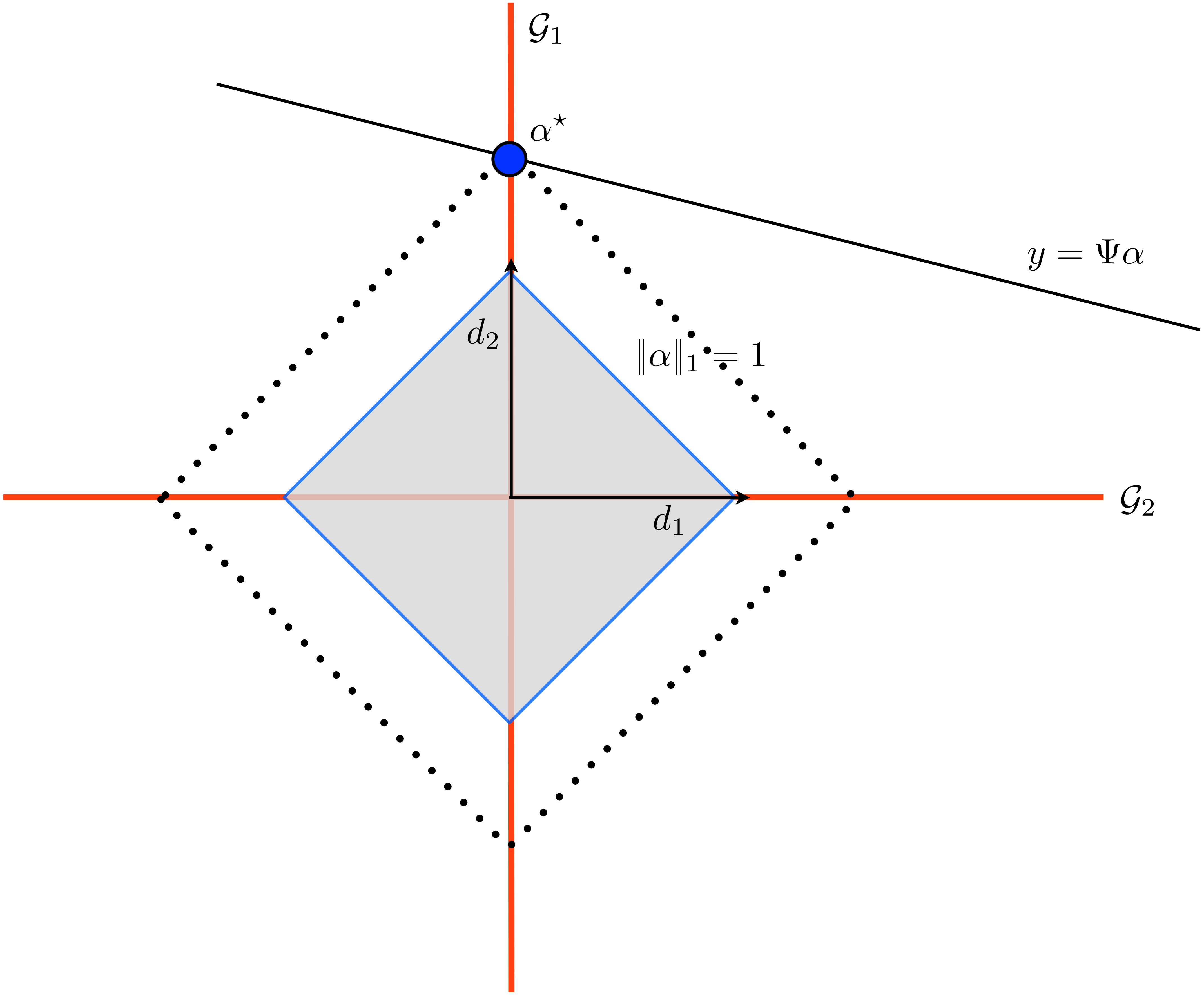


From Synthesis to Analysis Results



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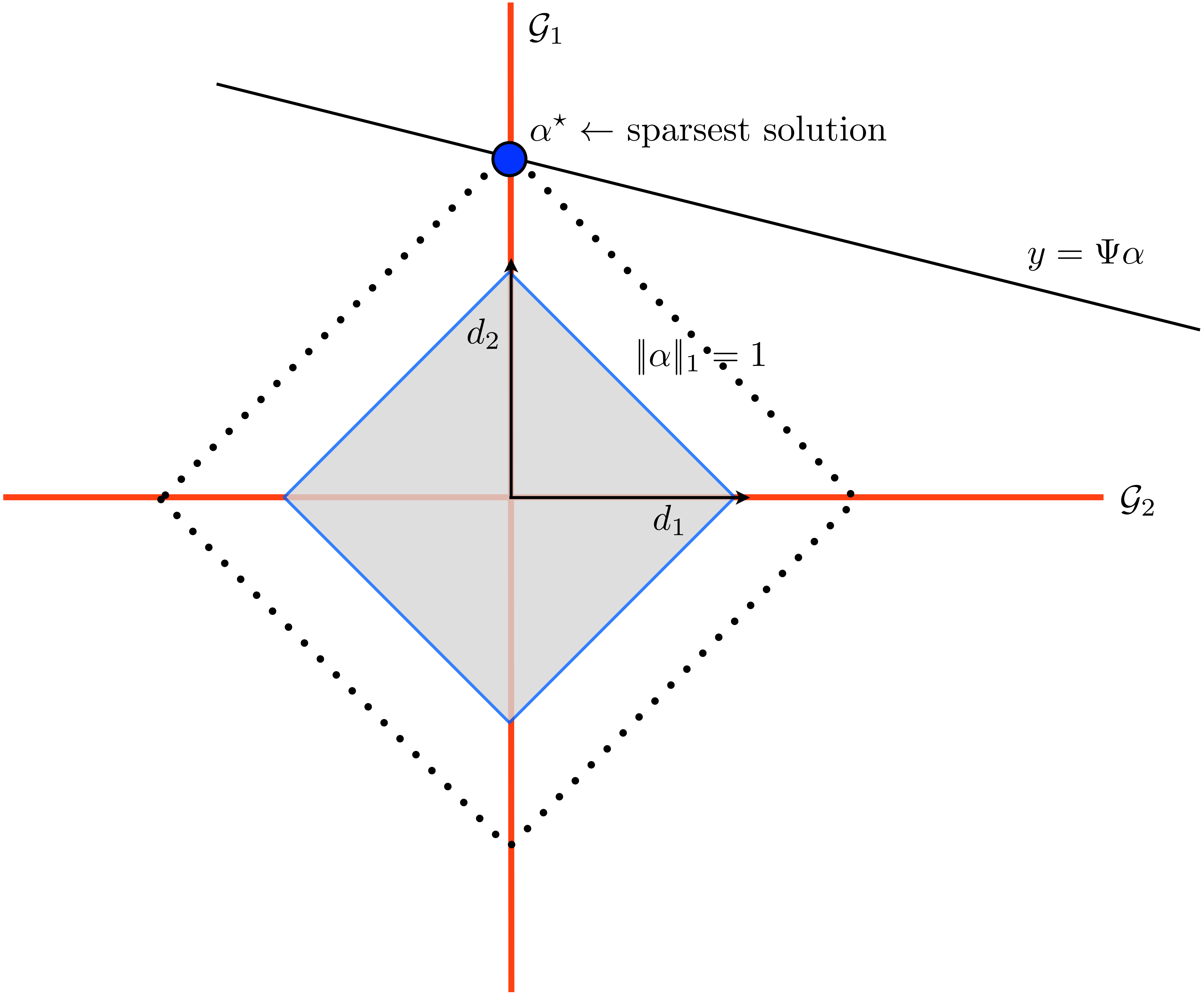

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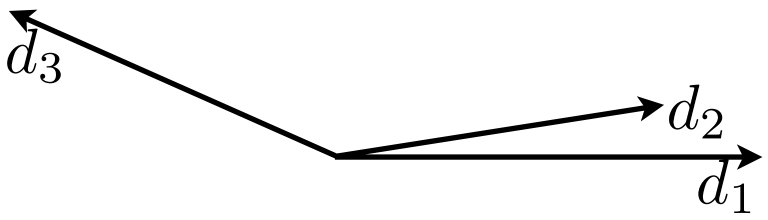

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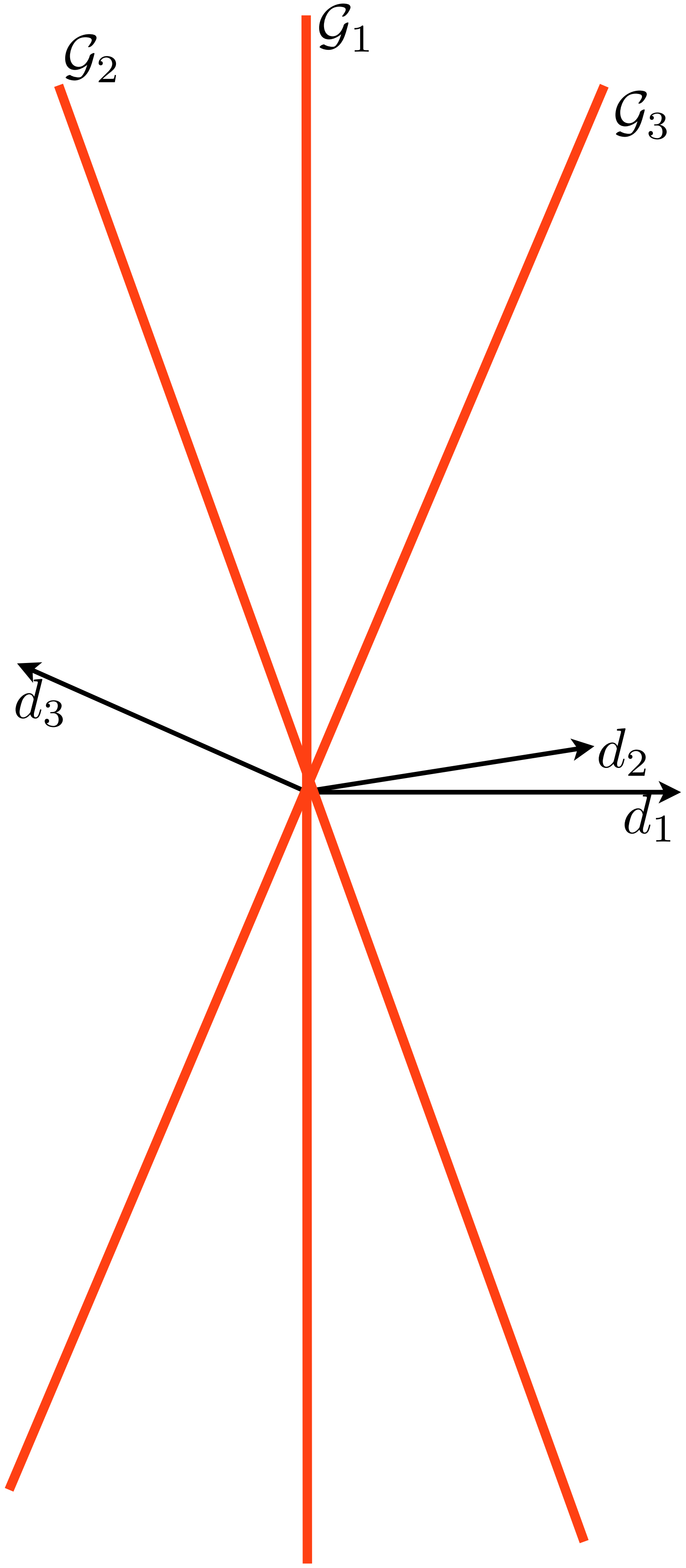

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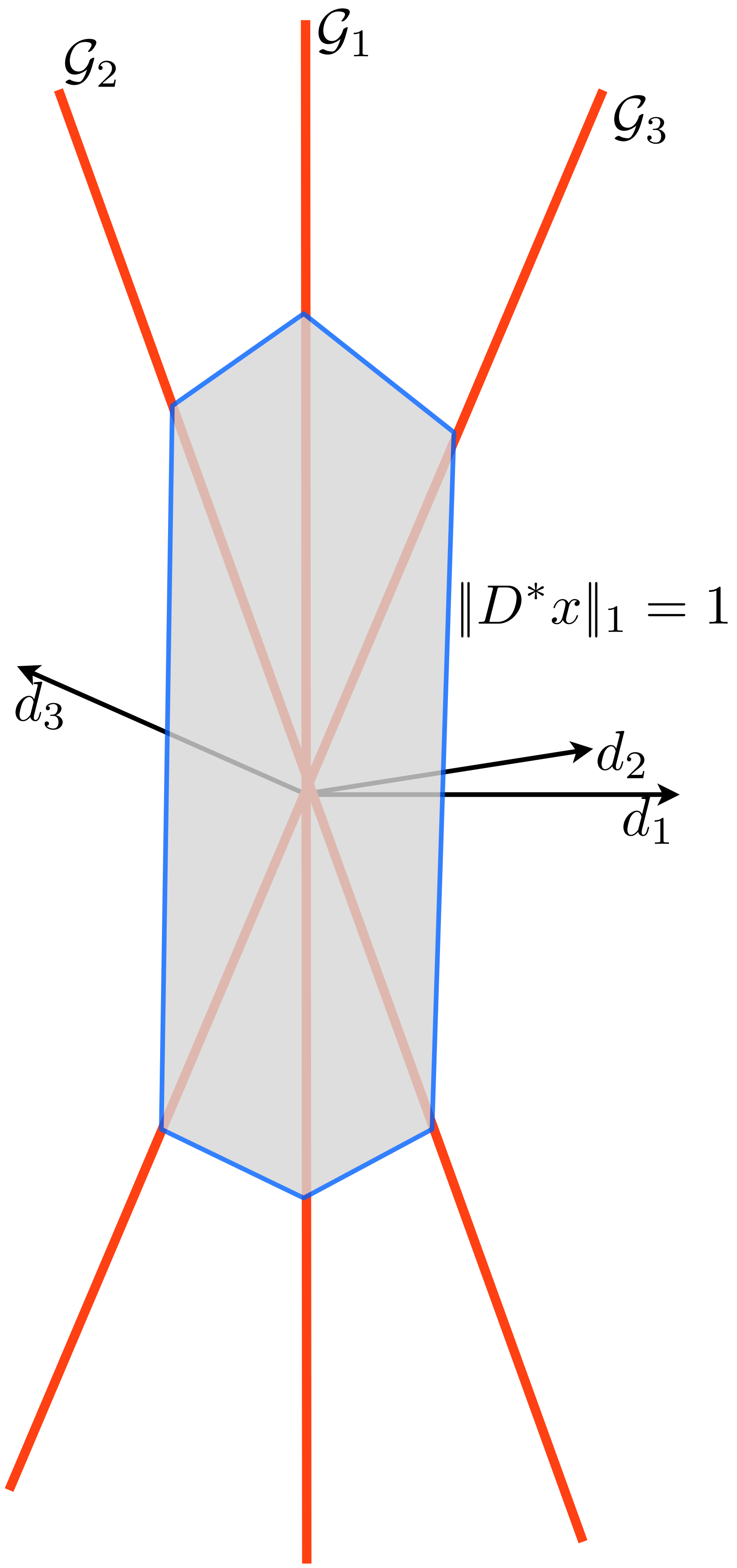

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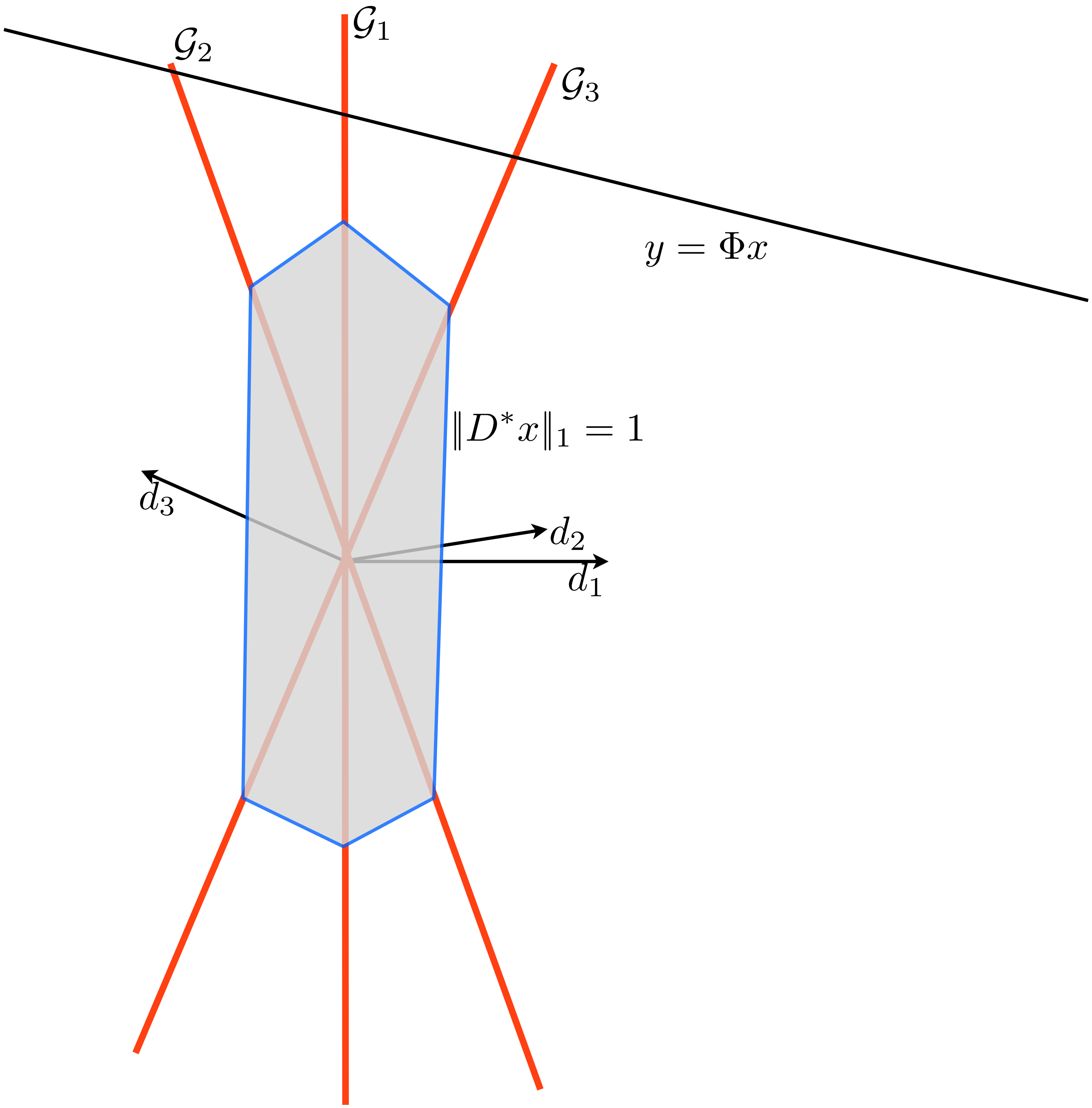

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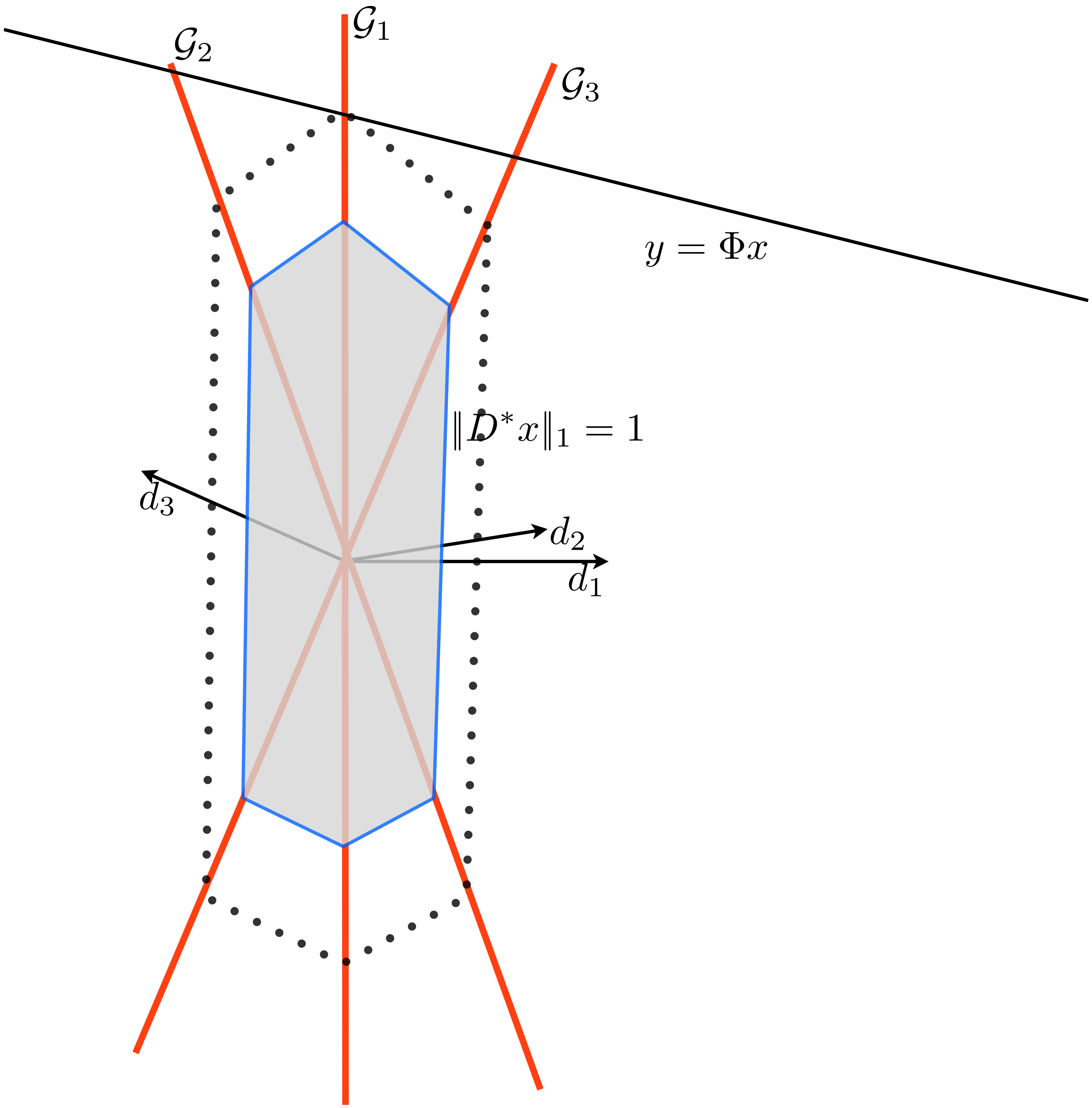

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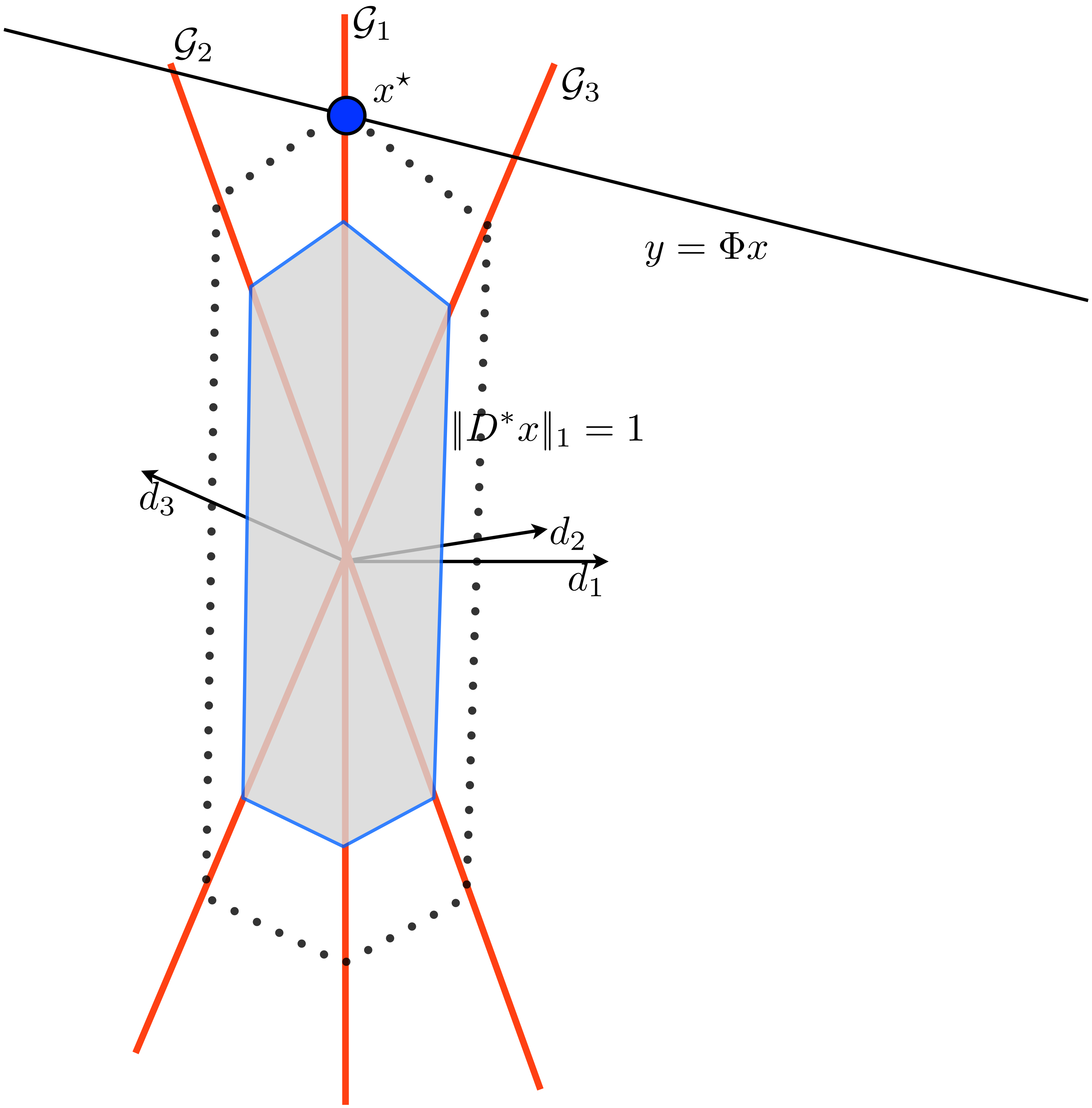

Overview

- Analysis vs. Synthesis Regularization
- Local Parameterization of Analysis Regularization
- Identifiability and Stability
- Numerical Evaluation
- Perspectives

Analysis is Piecewise Affine

Main idea: $\mathcal{G}_J$ is stable,

i.e solutions of $\mathcal{P}(y,\lambda)$ and $\mathcal{P}(y+\varepsilon,\lambda)$ lives in the same $\mathcal{G}_J$.

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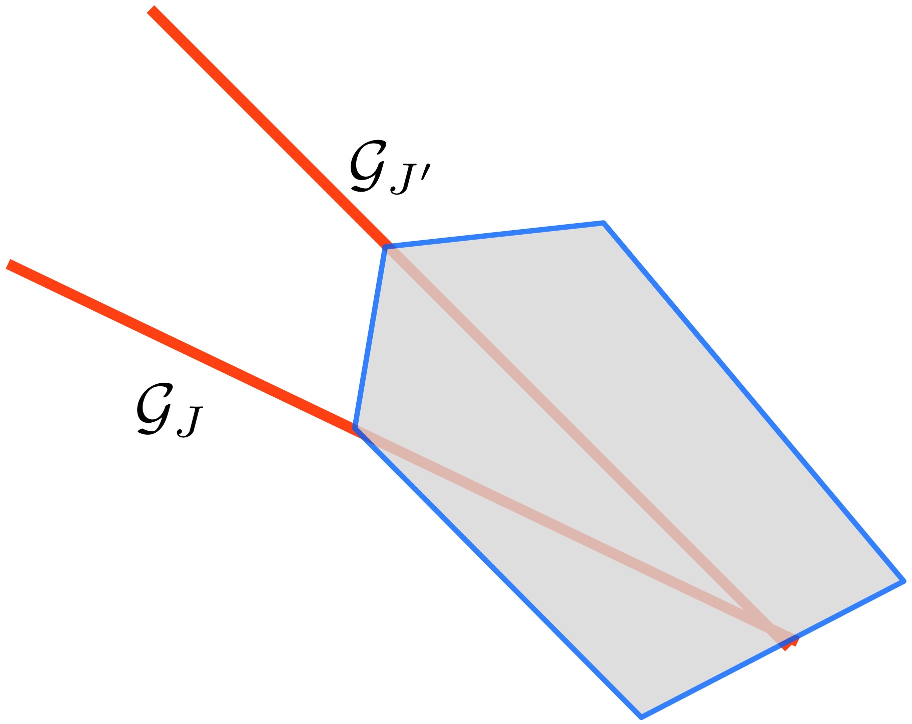

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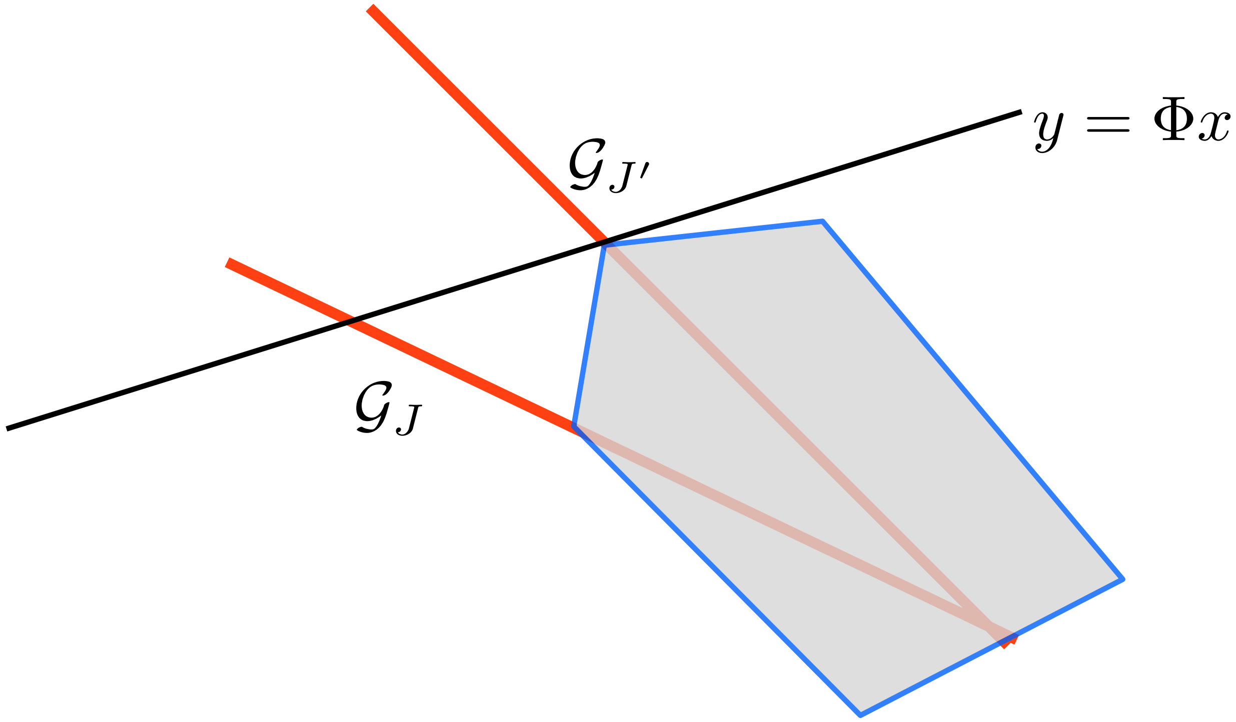

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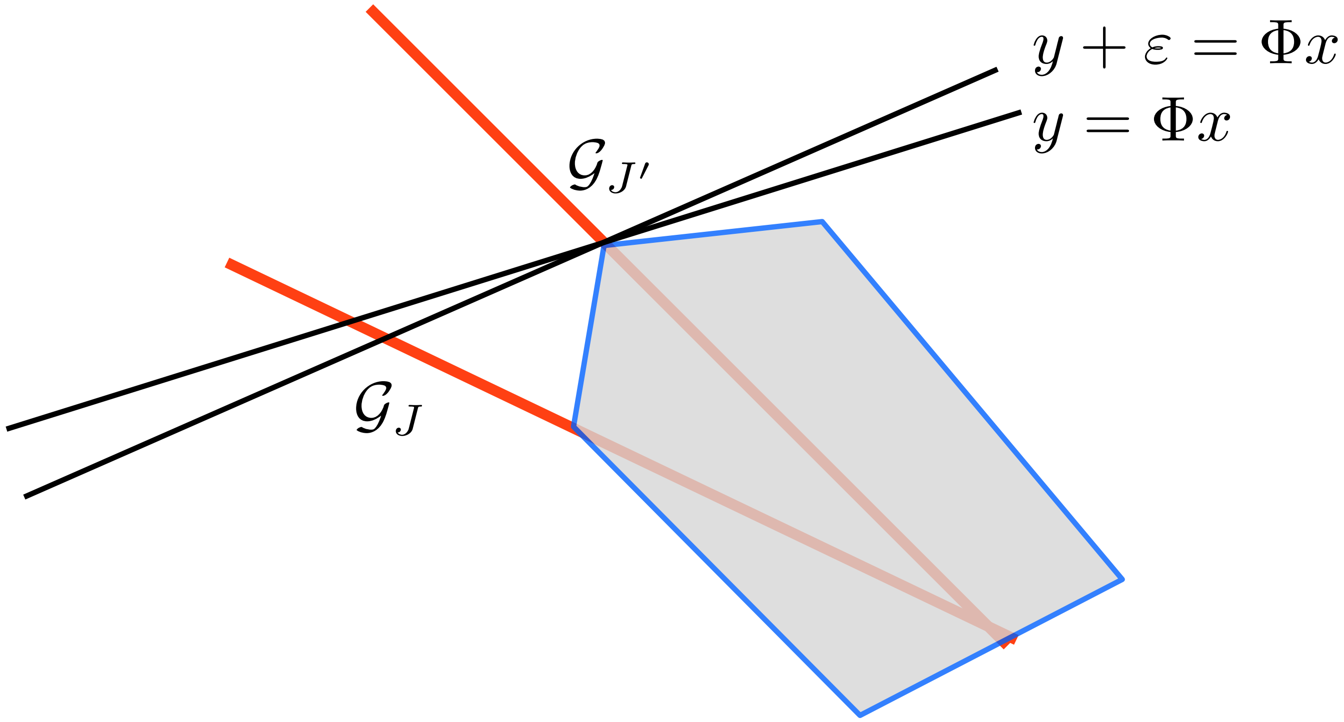

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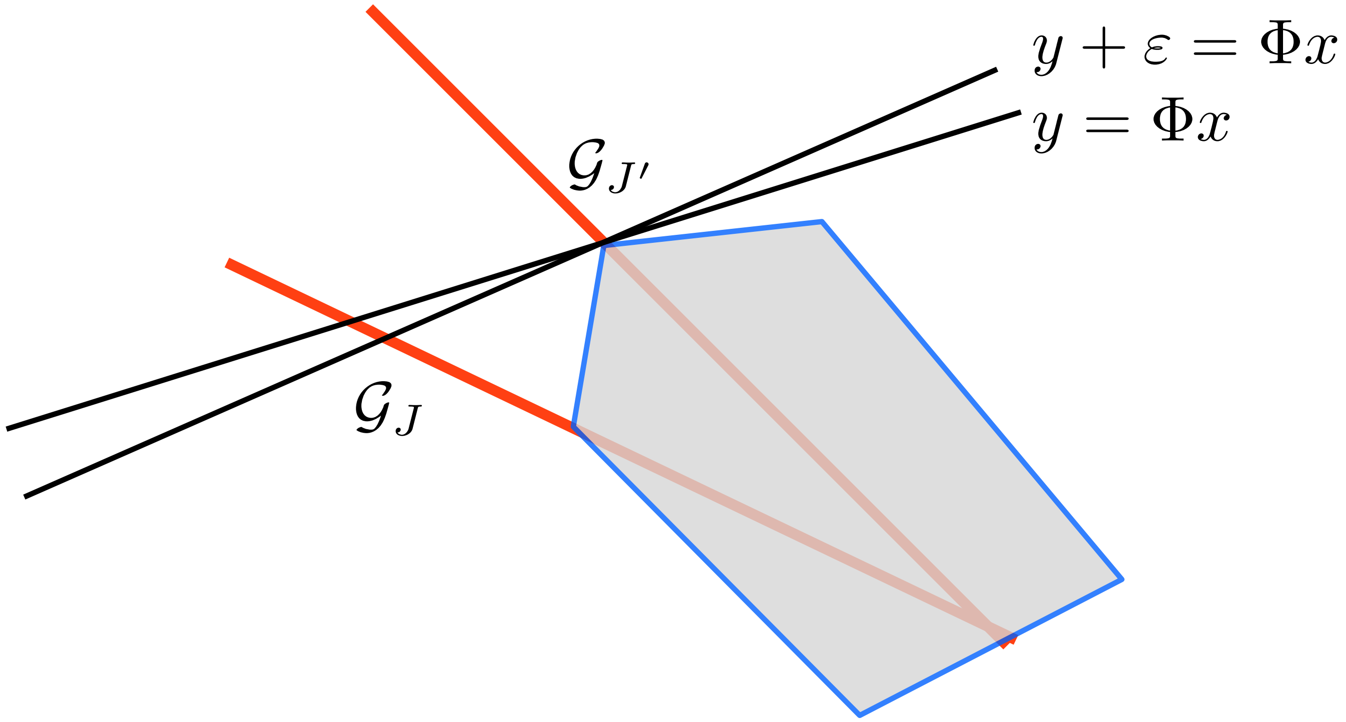

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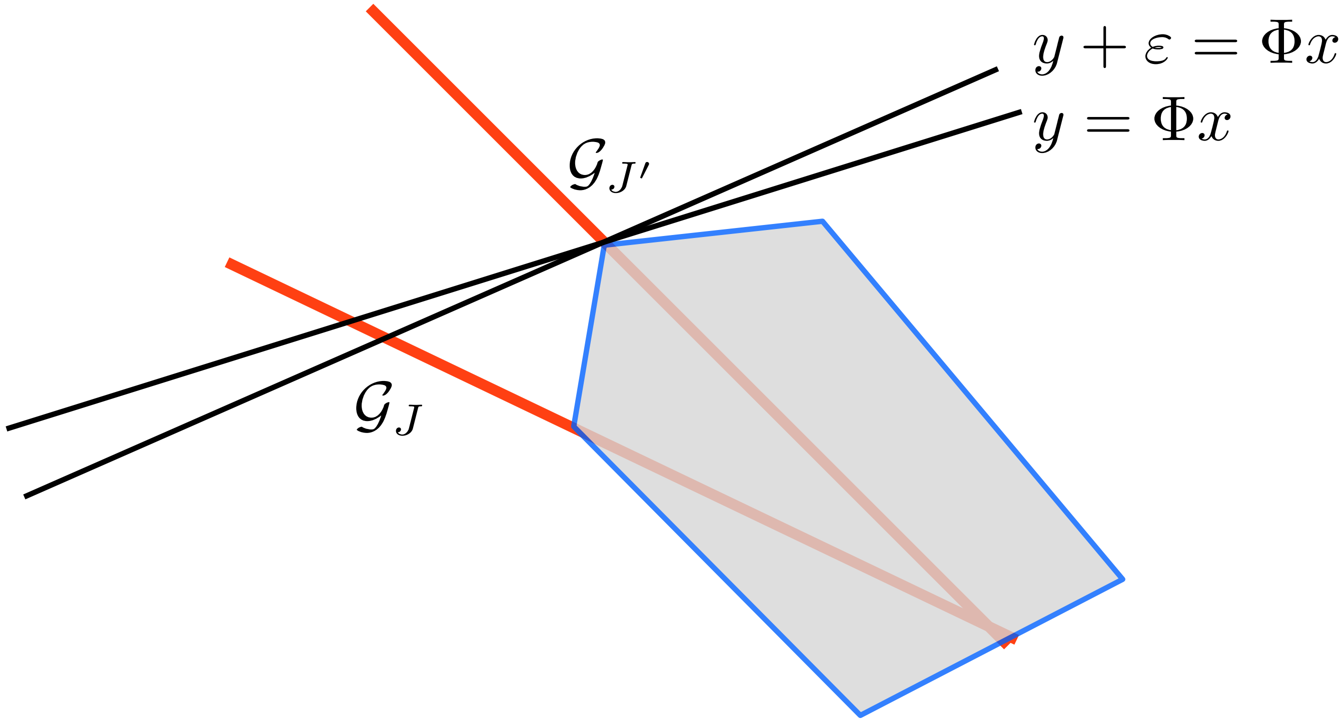

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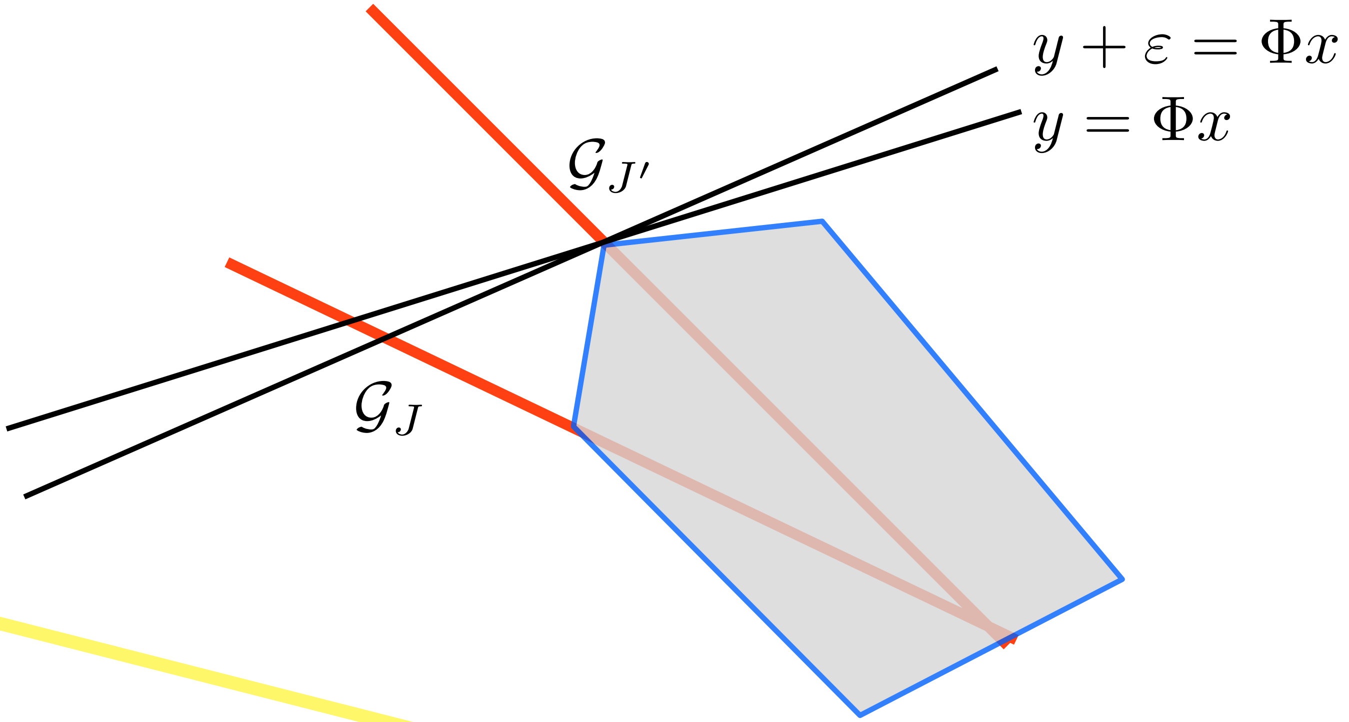

definition in few minutes

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Sketch of the Proof

Problem : Lasso

$$x^\star \in \underset{x \in \mathbb{R}^N}{\operatorname{argmin}} \; \frac{1}{2}\|y - \Phi x\|_2^2 + \lambda \|D^* x\|_1 \qquad \mathcal{P}(y, \lambda)$$

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— We fix observations y

— $I, J, s = \operatorname{sign}(D^* x^\star)$ are fixed by $x^\star$

First Order Conditions

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First-order conditions of Lasso

$$x^\star \text{ solution of } \mathcal{P}(y,\lambda) \Leftrightarrow \exists \sigma \in \Sigma_y(x^\star), \|\sigma\|_\infty \leqslant 1$$

$$\Sigma_y(x) = \left\{ \sigma \in \mathbb{R}^{|J|} \setminus \underbrace{\Phi^*(\Phi x - y)}_{\text{Gradient}} + \underbrace{\lambda D_I s + \lambda D_J \sigma}_{\text{Subdifferential}} = 0 \right\}$$

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$$x(y) \in \underset{x \in \mathcal{G}_J}{\operatorname{argmin}} \frac{1}{2}\|y - \Phi x\|_2^2 + \lambda \|D^* x\|_1$$

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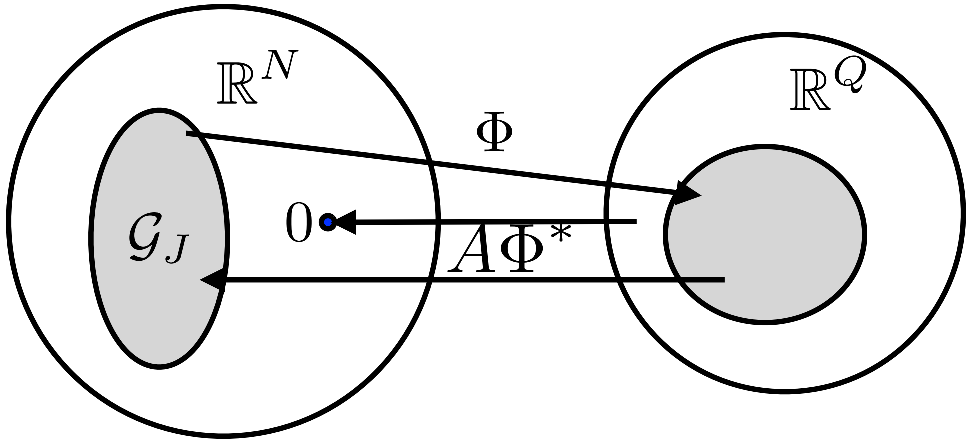

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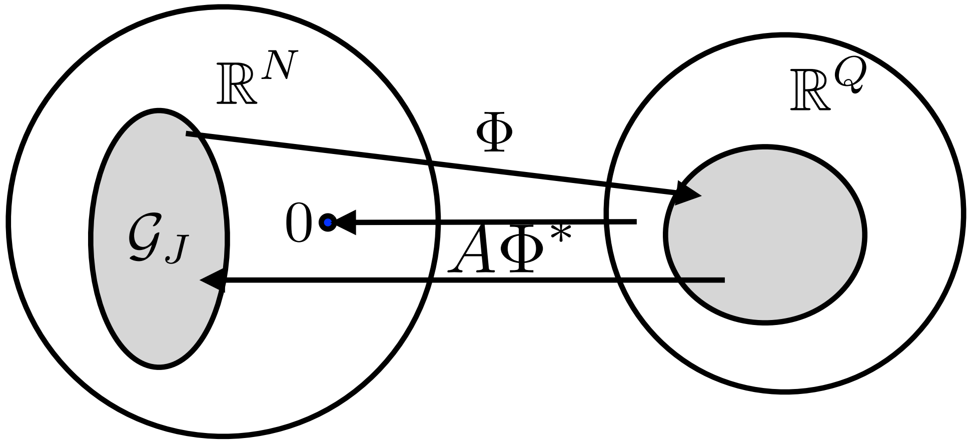

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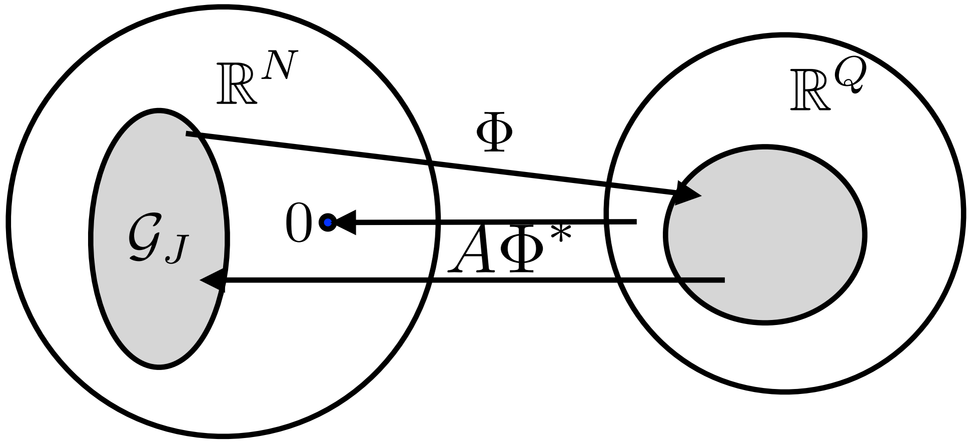

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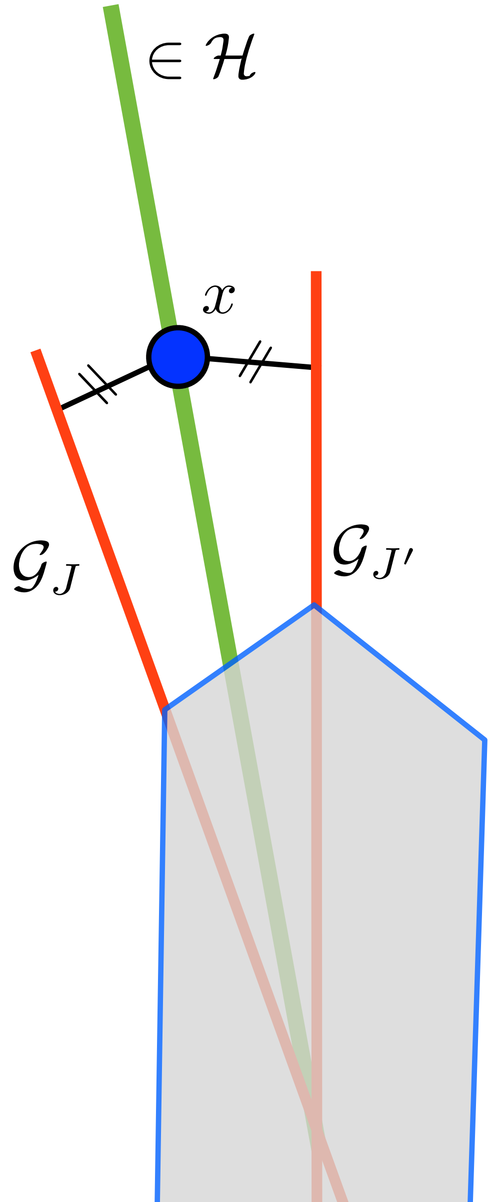

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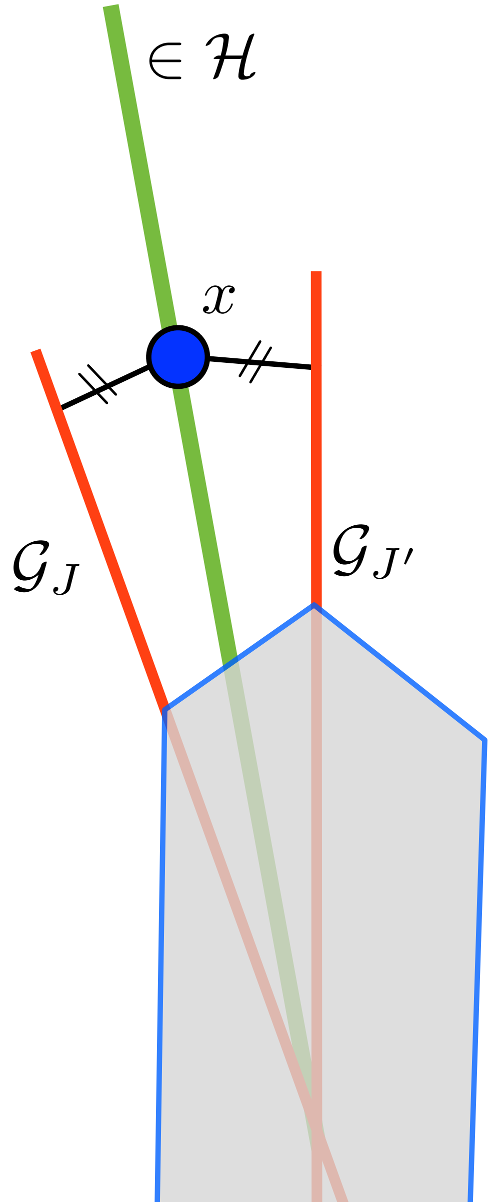
Open question

Smallest union of subspace containing $\mathcal{H}$?



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Sign stability

- Check that $x(\bar{y})$ is indeed solution of $\mathcal{P}(\bar{y}, \lambda)$

Use of first order conditions

Remember !

$$x(\bar{y}) = A\Phi^* \bar{y} - \lambda A D_I s$$

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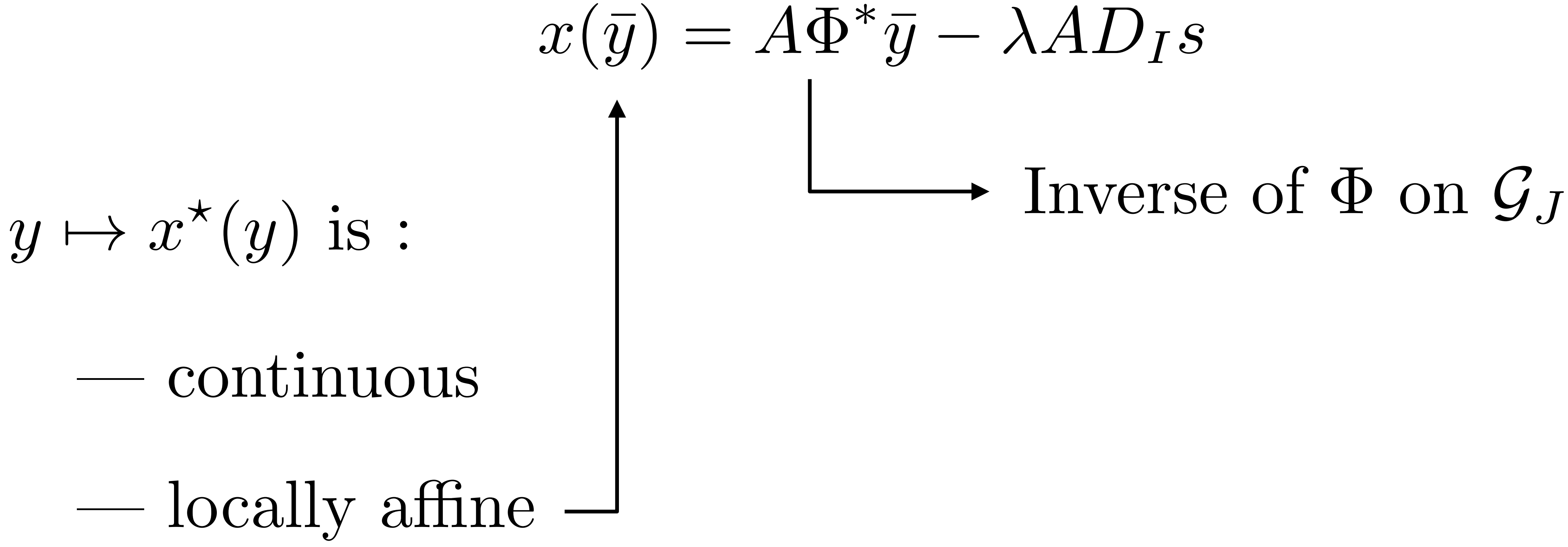

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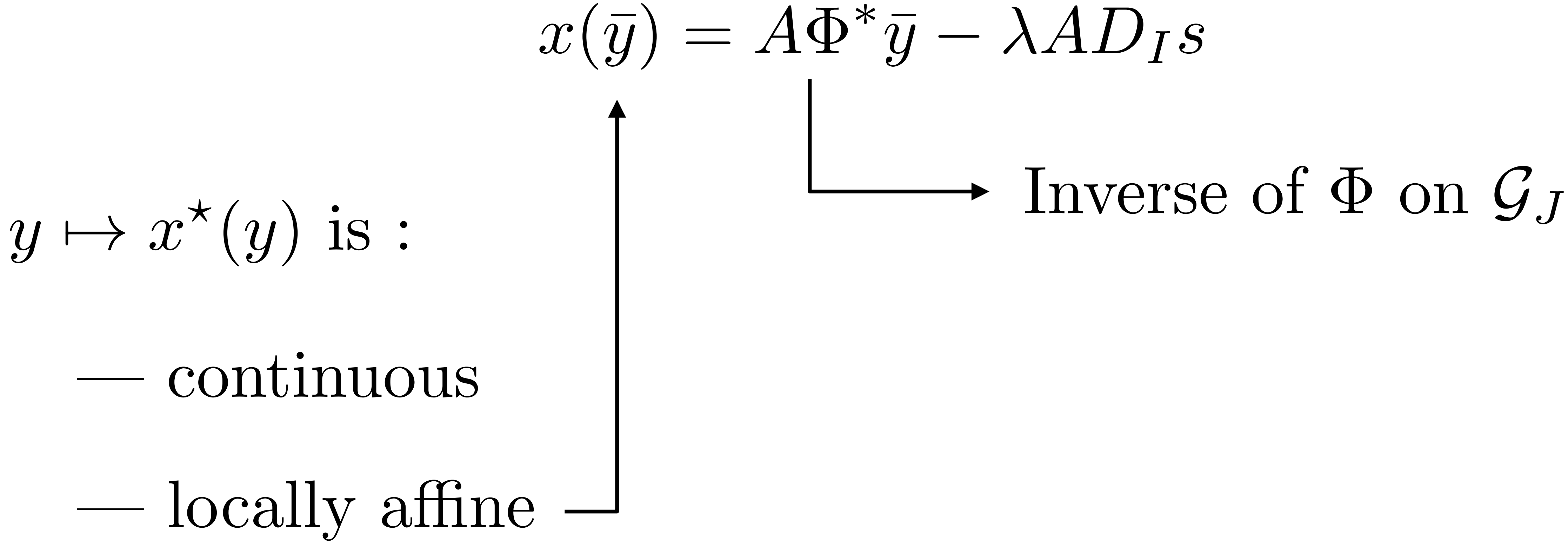

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Useful for :

— Robustness study

— SURE denoising risk estimation

— Inverse problem on Φx

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Strategy: $\mathcal{P}(y, \lambda)$ is almost $\mathcal{P}(y, 0)$ for small values of λ

Assumption: $\mathcal{G}_J$ must be stable for small values of λ

$\longrightarrow$ Restrictive condition !

But gives a stability results for small noise.

Noiseless and Sign Criterion

Algebraic criterion *on sign vector*

$$\Omega = D_J^+(\Phi^*\Phi A - \mathrm{Id})D_I$$

$$F(s) = \min_{w \in \mathrm{Ker}\, D_J} \|\Omega s - w\|_\infty$$

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Let $x_0 \in \mathbb{R}^N$ be a fixed vector, and $J = I^c$ where $I = I(D^* x_0)$.
Suppose that $\mathrm{Ker}\,\Phi \cap \mathcal{G}_J = \{0\}$.
If $F\left(\mathrm{sign}\left(D_I^* x_0\right)\right) < 1$ then x_0 is identifiable.

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Specializes to Fuchs results for synthesis ($D = \mathrm{Id}$)

Nam et al. Results

“Cosparse” model [Nam 2011]

Only other work on analysis recovery

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More intrinsic criterion

$\longrightarrow$ *But* no noise robustness, even for small ones

Sketch of the Proof

Idea: Study $\mathcal{P}(y, \lambda)$ for $\lambda \approx 0$

$$x_\lambda(\Phi x_0) = A\Phi^*\Phi x_0 - \lambda A D_I s$$

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$x_\lambda(\Phi x_0)$ solution of $\mathcal{P}(\lambda)$ and $x_\lambda(\Phi x_0) \xrightarrow[\lambda \to 0]{} x_0(\Phi x_0)$

$$\Downarrow$$

$$x_0(\Phi x_0) \text{ solution of } \mathcal{P}(0)$$

Sketch of the Proof

Idea: Study $\mathcal{P}(y, \lambda)$ for $\lambda \approx 0$

$$x_\lambda(\Phi x_0) = A\Phi^*\Phi x_0 - \lambda A D_I s$$

λ small enough to have $\text{sign}(D^* x_\lambda(\Phi x_0)) = \text{sign}(D^* x_0)$

$$\lim_{\lambda \to 0} x_\lambda(\Phi x_0) = A\Phi^*\Phi x_0 = x_0$$

$$x_\lambda(\Phi x_0) \text{ solution of } \mathcal{P}(\lambda) \text{ and } x_\lambda(\Phi x_0) \xrightarrow[\lambda \to 0]{} x_0(\Phi x_0)$$

$$\Downarrow$$

$$x_0(\Phi x_0) \text{ solution of } \mathcal{P}(0)$$

$$F(\text{sign}(D^* x_\lambda(\Phi x_0)) < 1 \Rightarrow x_\lambda(\Phi x_0) \text{ unique solution}$$

Small Noise Recovery

Suppose we observe $y = \Phi x_0 + w$

$$\text{Does } \underset{\Phi x = y}{\operatorname{argmin}} \|D^* x\|_1 \text{ recovers } x_0 + A\Phi^* w \text{ ?}$$

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Question: And for an arbitrary noise ?

Noisy and Support Criterion

Settings: $y = \Phi x_0 + w$, with w bounded noise.

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Theorem 3

Suppose $\mathrm{ARC}(I) < 1$ and $\lambda > K \dfrac{\|w\|}{1 - \mathrm{ARC}(I)}$ then $x_\lambda(y)$ is the unique solution of $\mathcal{P}(y, \lambda)$ and

$$\|x_\lambda(\bar{y}) - x_0\| = O(\lambda)$$

Remember !

$F(s) = \min_{w \in \text{Ker}\, D_J} \|\Omega s - w\|_\infty$	$\text{ARC}(I) = \max_{x \in \mathcal{G}_J} F(\text{sign}(D_I^* x))$
sign	support
Noiseless	Noisy
Vector identifiability	Support identifiability

From Theory to Numerics

We give a sufficient condition for identifiability.

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We give a sufficient condition for identifiability.

How far are we from a necessary condition ?

Overview

- Analysis vs. Synthesis Regularization
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Proximal Operator

f l.s.c convex function from C convex of an Hilbert $\mathbb{H}$ in $\mathbb{R}$.

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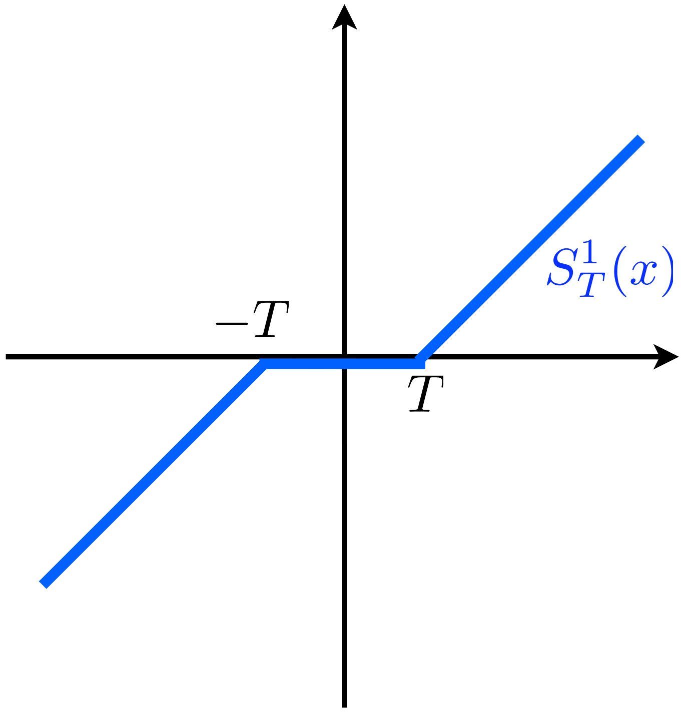
Fundamental examples:

$$\operatorname{prox}_{\|\cdot\|_1} = S_T^1.$$

$$\operatorname{prox}_{i_C} = P_C$$



How to Solve These Regularizations ?

Primal-dual schemes

$$\min_{x \in \mathbb{R}^N} \mathcal{L}(K(x)) \quad \text{where} \quad \left\{ \begin{array}{l} \mathcal{L}(g,u) = \frac{1}{2}\|y-g\|^2 + \lambda \|u\|_1 \\ K(x) = (\Phi x, D^* x) \end{array} \right.$$

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Alternating Direction Method of Multipliers [Chambolle, Pock]

$$\begin{aligned} u_n &= \operatorname{prox}_{\sigma \mathcal{L}^*}(u_{n-1} + \sigma K(z_{n-1})) \\ x_n &= \operatorname{prox}_{\tau G}(x_{n-1} - \tau K^*(u_n)) \\ z_n &= x_n + \theta(x_n - x_{n-1}) \end{aligned}$$

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For $\mathcal{P}(y,0)$, $\|y-g\|^2 \rightarrow i_{\{y\}}$

Computing Criterions

Unconstrained formulation

$$\begin{tikzcd} F(s) = \min\limits_{w\in\mathbb{R}^N} & \|\Omega s - w\|_\infty \arrow[d] & + & i_{\mathcal{D}}(w) \arrow[d] \\ & \mathrm{P}_{\|\cdot\|_1=1} & \text{Prox} & \mathrm{P}_{\mathcal{D}} \end{tikzcd}$$

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ARC difficult to compute (non-convex)

$$\begin{array}{rrll} & \mathrm{ARC}(I) & = \max_{x\in\mathcal{G}_J} F(\mathrm{sign}(D_I^* x)) & \text{non-convex} \\ \leqslant & \mathrm{wARC}(I) & = \max_{s\in\{-1,1\}^{|J|}} F(s) & \text{non-convex} \\ \leqslant & \mathrm{oARC}(I) & = \|\Omega\|_{\infty\to\infty} & \text{easy} \end{array}$$

More on Signal Models

Signal model : *"Union of subspace"*

$$\Theta = \bigcup_{k \in \{1 \ldots P\}} \Theta_k \quad \text{where} \quad \Theta_k = \{\mathcal{G}_J \setminus \dim \mathcal{G}_J = k\}$$

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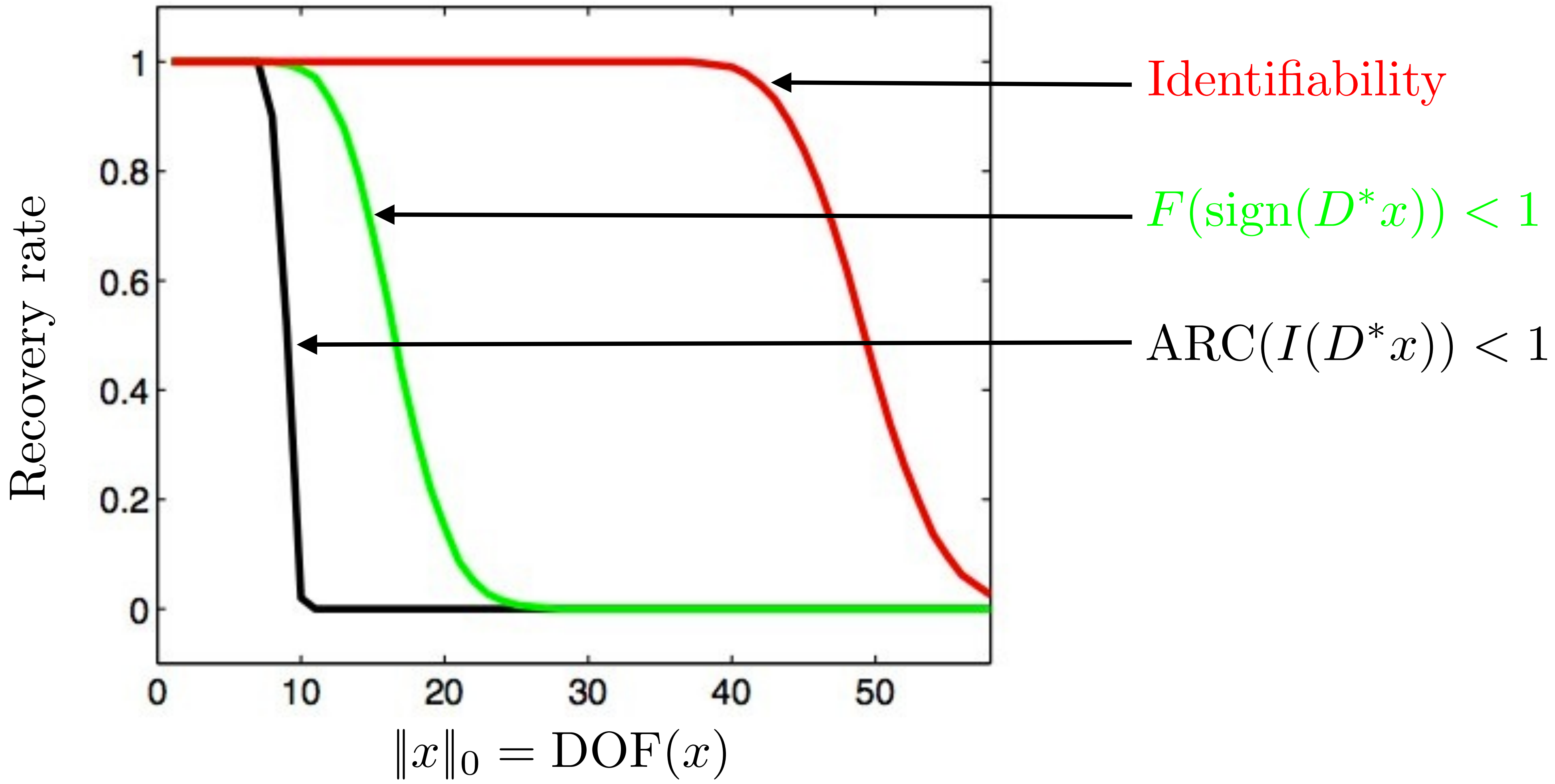
Good one : $\mathrm{DOF}(x) = \dim \mathcal{G}_J$

Random Settings

1) Synthesis results

Compressed sensing : $Q \ll N$



Credit to C. Dossal

Random Settings

2) Analysis results $\qquad D, \Phi$ Gaussian i.i.d random matrices

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Many dependancies between columns

$\longrightarrow$ Strong unstability

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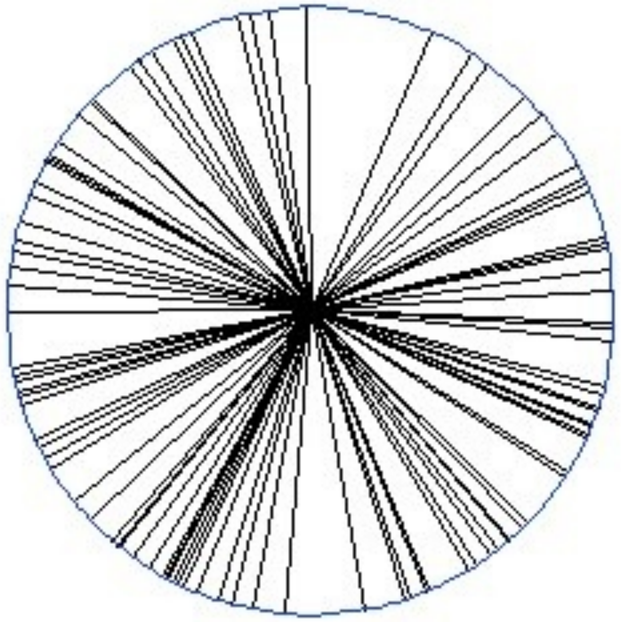

Close to ℓ_2 ball !

Limits : TV Instability

$D^* = \nabla, \Phi = \mathrm{Id}$

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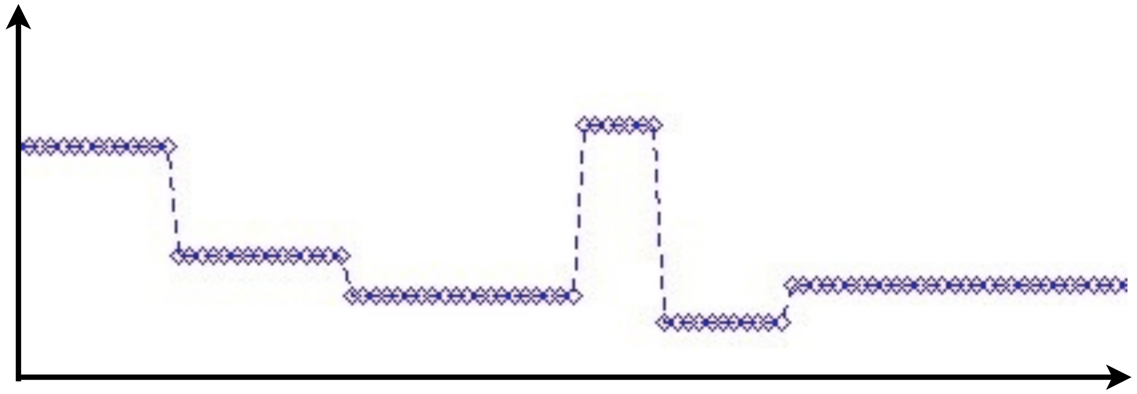
Θ_k : piecewise constant signals with $k-1$ step.



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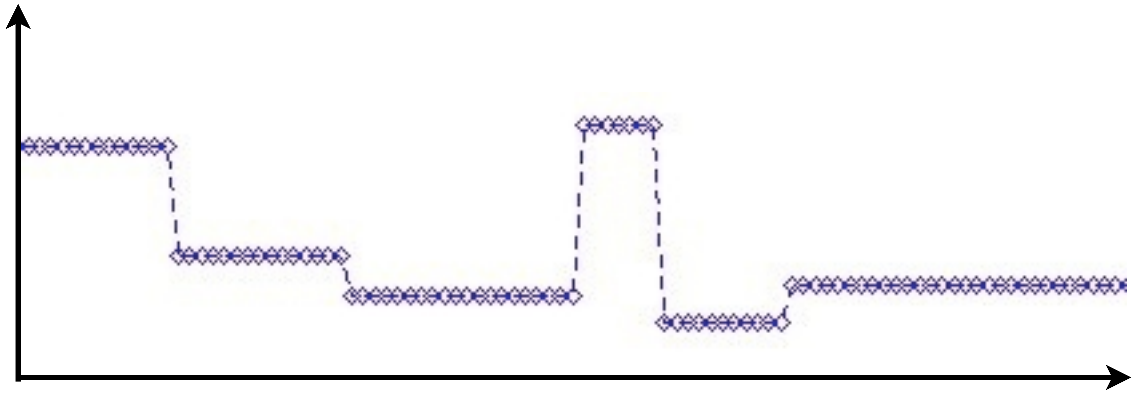
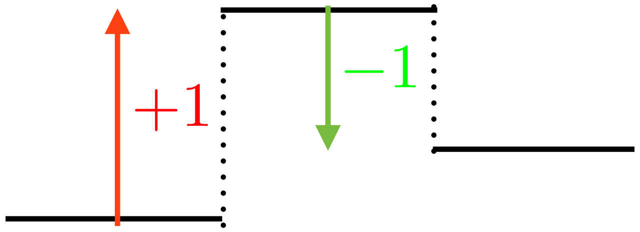


“*Box*”



$$F(s) = 1 - \varepsilon$$

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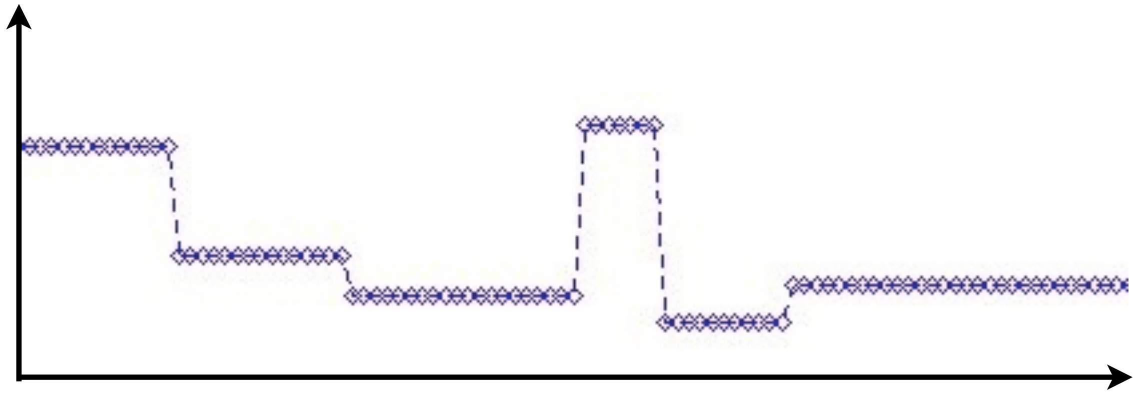

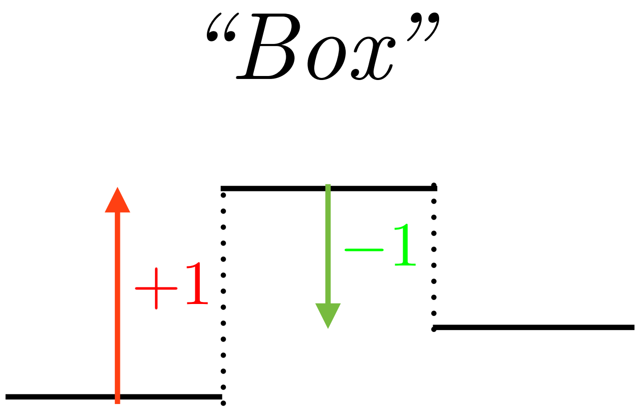

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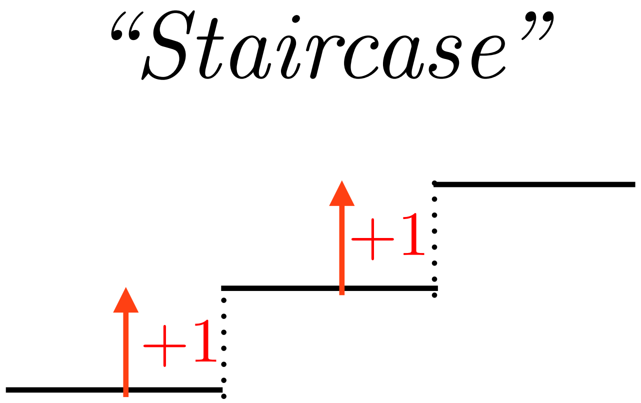

$$F(s) = 1$$

No noise stability
even for small one

Fused Lasso

$$\underset{x \in \mathbb{R}^N}{\operatorname{argmin}} \ \frac{1}{2}\|y - \Phi x\|_2^2 \quad \text{subject to} \quad \left\{ \begin{array}{l} \|\nabla x\|_1 \leqslant s_1 \\ \|x\|_1 \leqslant s_2 \end{array} \right.$$

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Signal Model: Characteristic functions sum

$$\Theta_2 : x_0 = \mathbf{1}_{[a,b]} + \mathbf{1}_{[c,d]}$$

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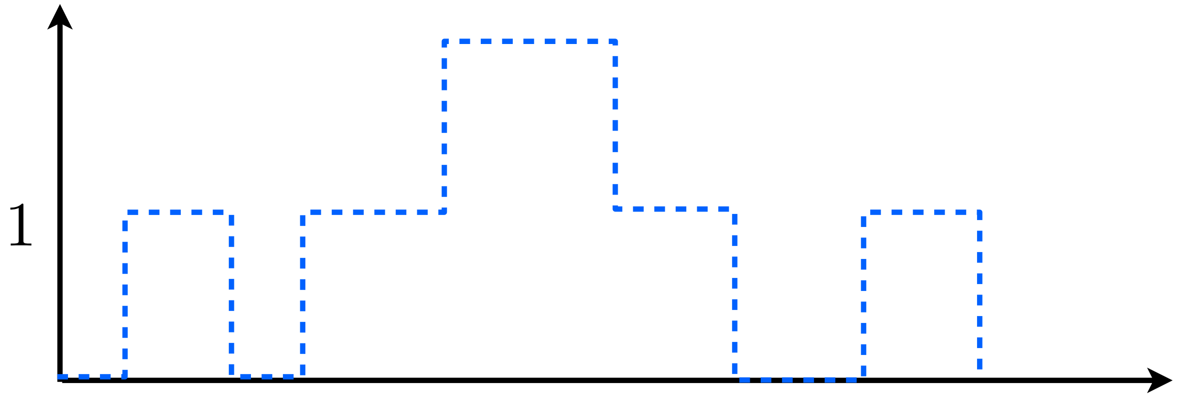
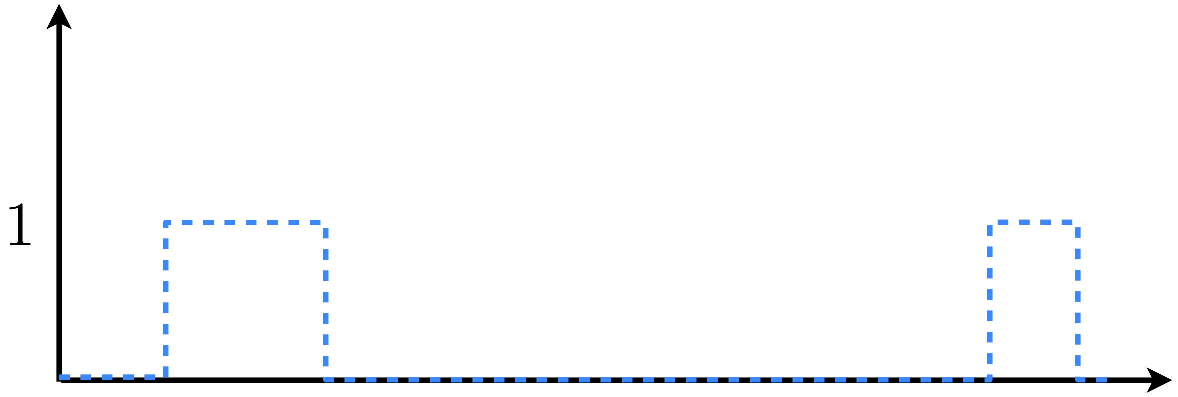


No overlap



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$$[a, b] \cap [c, d] \neq \emptyset \Rightarrow F(x_0) \geqslant 1$$

no noise robustness

Fused Lasso

Ω_{DIF}

$\varepsilon \mathrm{Id}$

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$[a, b] \cap [c, d] = \emptyset \Rightarrow$ 2 situations

Fused Lasso

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$$|c - b| \leqslant \xi(\varepsilon)$$

$$F(\operatorname{sign}(D^* x_0)) \geqslant 1$$

no noise robustness

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$\|c-b\| \leqslant \xi(\varepsilon)$	$\|c-b\| > \xi(\varepsilon)$
$F(\text{sign}(D^* x_0)) \geqslant 1$	$F(\text{sign}(D^* x_0)) = \text{ARC}(I) < 1$
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$|c - b| > \xi(\varepsilon)$

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strong noise robustness

Haar : similar results

Take-Away Messages

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— Analysis regularization *is* robust

— Geometry (union of subspaces) : key concept for recovery

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Deterministic theorem → treat the noise as a random variable

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Almost equal support recovery

Thanks

Joint work with

- Gabriel Peyré (CEREMADE, Dauphine)
- Charles Dossal (IMB, Bordeaux I)
- Jalal Fadili (GREYC, ENSICAEN)

Any questions ?

An Affine Implicit Mapping

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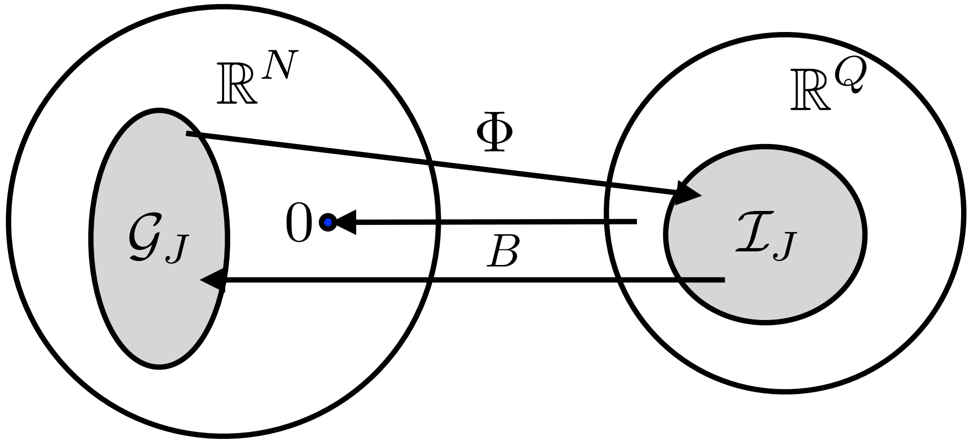
$$\mathcal{G}_J \overset{B}{\cong} \mathcal{I}_J = \Phi(\mathcal{G}_J)$$



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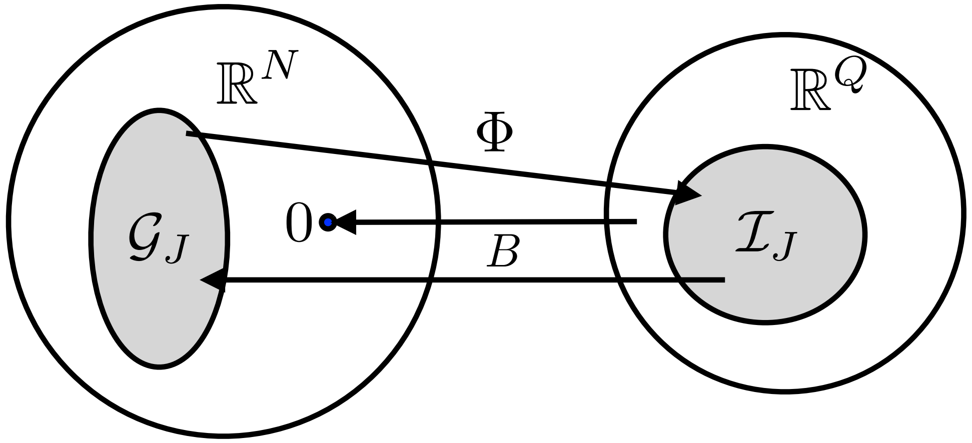

$$B : \begin{cases} B_{|\mathcal{I}_J} = \left(\Phi_{|\mathcal{G}_J}\right)^{-1} \\ B_{|\mathcal{I}_J^\perp} = 0 \end{cases}$$

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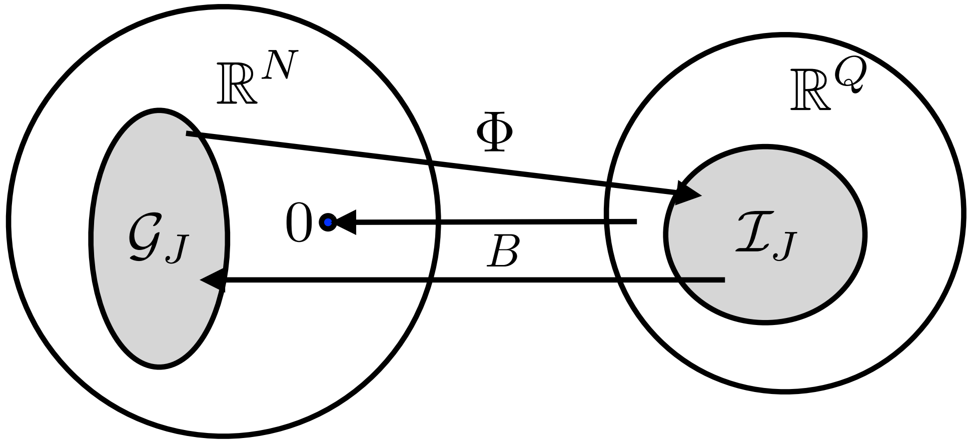

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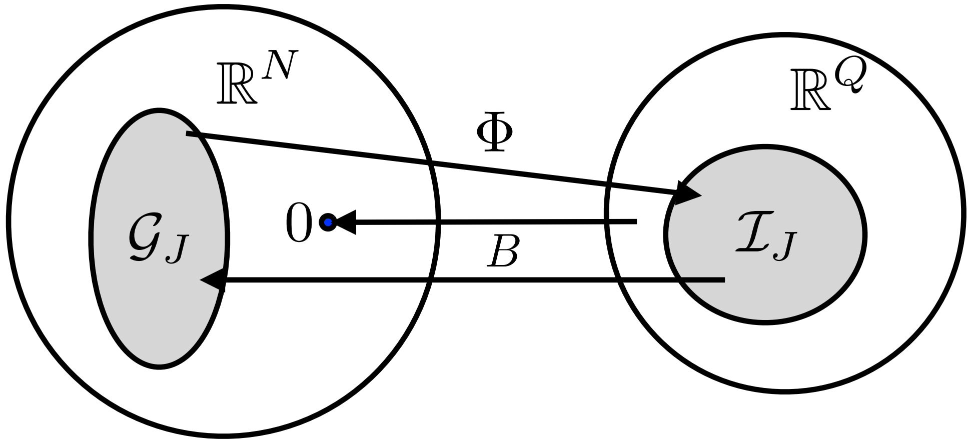

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$$C \begin{pmatrix} y \\ \mu \end{pmatrix} = \begin{pmatrix} \Phi x \\ 0 \end{pmatrix} \quad \text{where} \quad C = \begin{pmatrix} \Phi^*\Phi & D \\ D^* & 0 \end{pmatrix}$$